FORMAL METHODS FOR DERIVING GREEN-TYPE TRANSITIONAL AND UNIFORM ASYMPTOTIC EXPANSIONS FROM DIFFERENTIAL EQUATIONS

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A Thesis Submitted for the Degree of PhD
at the
University of St Andrews

1965

FORMAL METHODS FOR DERIVING GREEN-TYPE, TRANSITIONAL AND UNIFORM ASYMPTOTIC EXPANSIONS FROM DIFFERENTIAL EQUATIONS

A Thesis

presented by

Siebe Jorna, M.Sc.

to the

University of St. Andrews

in application for the Degree

of Doctor of Philosophy

DECLARATION

I hereby declare that the following Thesis is based on research carried out by me, that the Thesis is my own composition, and that it has not previously been presented for a Higher Degree.

CERTIFICATE

I certify that Siebe Jorna, M.Sc., has spent nine terms as a research student in the Department of Theoretical Physics of the United College of St. Salvator and St. Leonard in the University of St. Andrews, that he has fulfilled the conditions of Ordinance No. 16 of the University Court of St. Andrews and that he is qualified to submit the accompanying Thesis in application for the Degree of Doctor of Philosophy.

Research Supervisor

CAREER

From 1956 to 1960, I studied at the University of Western Australia and followed a course leading to the degree of Bachelor of Science with First Class Honours in Physics. During the first three years I received financial support in the form of a Commonwealth Scholarship, while in the fourth year I held a C.S.I.R.O. Junior Fellowship. In 1960, I was awarded an Australian Commonwealth Postgraduate Scholarship and began research for the degree of Master of Science in Theoretical Physics. Subsequently, tenure of a Canadian Commonwealth Postgraduate Scholarship enabled me to study Medical Physics at the University of Toronto during the academic year 1960-1961. In October 1961, I accepted a D.S.I.R. Research Assistantship tenable in the Department of Theoretical Physics in St. Salvator's College at the University of St. Andrews. Since that time, I have been engaged in research on the subject of this Thesis. In 1962, I was awarded the degree of Master of Science (Western Australia) in Theoretical Physics. From October 1962, I have held the post of Research Assistant ~~Lecturer~~ in the Department of Theoretical Physics at the University of St. Andrews.

ACKNOWLEDGEMENTS

I am pleased to acknowledge my obligation to my supervisor, Professor R.B. Dingle, for suggesting the problems treated in this Thesis, and for much generous advice on all aspects of the work.

I also wish to thank Mr. Robert Ferrier for assisting in the development of the computer programme recorded in § 8.71.

Finally, I am grateful to my wife for typing this Thesis from a manuscript which at times was almost legible.

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1. INTRODUCTION AND SUMMARY.

On expanding a relatively complicated function, $f(z)$ say, in powers of z, two distinctly different types of expansion may be encountered: one convergent and one semi-convergent or asymptotic. The rigorous mathematical features of asymptotic expansions have already been discussed widely (cf. e.g. Watson 1944, Erdélyi 1956, Jeffreys 1962, Copson 1965) and in this introduction only those formal aspects of this topic will be outlined which are closely related to the subject of this thesis.

From a computational point of view, the convergent expansion remains useful until $|z|$ becomes very large, i.e. until the circle of convergence is approached. Then a prohibitively large number of terms may have to be calculated in order to obtain only a rough value of $f(z)$. On the other hand, for sufficiently large $|z|$, even the first term in the asymptotic expansion normally provides a reasonable approximation to the function. Experience shows, moreover, that the approximation techniques applied in physics - e.g. perturbation and WKB methods - commonly lead to expansions of the asymptotic rather than the convergent type. It is thus not surprising that the theory of asymptotic expansions has played an important role in a diversity of physical problems. Recently, asymptotic expansions have found application in such widely varying fields as, for instance[+]: elastic shell problems, motion in a stratified atmosphere, stability of boundary layers in a compressible gas, spiral structure of disk galaxies, resonance in elliptic cylinders (see also § 8.1 in this thesis), diffraction by spheres and spheroids (see also § 9.2), pressure waves from nuclear explosions, Regge poles and tunnelling through high periodic barriers (Das 1957, Stejskal and Gutowski 1958).

Characteristically, the first few terms in an asymptotic expansion form a rapidly decreasing sequence, but then they start to increase. In early investigations it was thought that the divergent part of the asymptotic expansion was meaningless, and that the maximum accuracy was fixed by the least term. Airey (1937), however, introduced the concept of converging factors, and showed

[+]Where not given, references are as quoted by Clark, Lin and Kazarinoff in recent symposium proceedings edited by Wilcox (1964).

that when the terms are not all of the same sign, the divergent part of the asymptotic expansion can be split up into summable series. A refinement was devised by Miller (1952) for Weber parabolic cylinder functions. He interpreted the divergent part as the difference between the function and the convergent part of the asymptotic expansion, and then obtained a new expansion from the inhomogeneous differential equation satisfied by this difference. Classifying asymptotic expansions by the general form of the later terms into four basic groups, Dingle (1958 a,b,c; 1959 a,b,c) broadened the scope of Airey's method considerably by interpreting the remainder - not necessarily just the divergent part - of truncated asymptotic expansions in the Borel sense (1899). He published extensive tables of so-called 'basic converging factors', and subsequently illustrated their use by detailed calculations for a wide range of functions, including, for example, error, Fresnel, exponential, sine, cosine, Fermi-Dirac and Bose-Einstein integrals, gamma and polygamma functions, confluent hypergeometric, parabolic cylinder and Bessel functions.

In the above, reference has mainly been made to asymptotic expansions which are either of the Stokes- or the Green-type. Stokes expansions are asymptotic expansions essentially in inverse powers of the independent variable, z say, and any parameter is considered to be small. Green-type expansions are valid when both variable and parameter can be large. There are, however, two further important types of asymptotic expansion, namely transitional and uniform expansions. Green-type expansions fail for certain combinations of variable and parameter; expansions covering these cases will be termed transitional expansions. We shall generally obtain such transitional expansions from the much more general uniform expansions, which are valid for large parameter but hold uniformly with respect to z. As will be shown, it is also always possible to regain the corresponding Green-type expansions from the uniform expansions.

In the present work, we develop and illustrate powerful, but straightforward, formal methods for deriving asymptotic expansions from differential equations. In the second chapter, the 'inverse Frobenius method' for deriving Stokes expansions is exemplified. The main body of this thesis, however , consists of the development of the new Green-Liouville-Mellin transform method, and its detailed application to modified Bessel functions, parabolic cylinder functions, Whittaker functions, Poiseuille functions, confluent hypergeometric functions, and also to periodic Mathieu functions and oblate spheroidal wave functions, all with at least one parameter large[+]. The wide scope of the method is evinced by the fact that treatment of the essentially eigenvalue problem posed by the two last-named cases requires no additional techniques. This method, as will be explained in detail in chapter 3, yields Green-type, transitional and uniform expansions.

The transitional expansions found in this way are usually of a simpler form than those derived by alternative processes (e.g. perturbation theory). To state an example, the asymptotic expansions for the periodic Mathieu functions $ce(z,h)$ and $se(z,h)$ valid near $|z| = \frac{1}{2}\pi$ that have been obtained in earlier work contain the <u>complicated parabolic cylinder functions</u> (cf. Meixner 1948, Sips 1949, Dingle and Müller 1962). By contrast, our methods yield expansions of comparable

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[+]Some of this work has already been published (cf. Jorna (1964a,b;1965)).

applicability, but involving <u>only elementary functions</u>. To demonstrate their usefulness, we have fed these expansions into a digital computer and obtained extensive tables for $ce(z,h)$ and $se(z,h)$ in the range $50^\circ \leq z \leq 90^\circ$. Extracts from these tables and comparisons with correct results are given in § 8.71.

Following the chapters on the introduction and applications of the Mellin transform technique, there is some preliminary work on a new method for determining the general term in Green-type expansions. The method is illustrated by detailed calculations for modified Bessel and parabolic cylinder functions.

In the final chapter, we present certain suggestions for further work.

2. ASYMPTOTIC EXPANSIONS BY THE INVERSE FROBENIUS METHOD.

Many differential equations of the type

$$Ly(z) = 0, \tag{1}$$

where L is a second-order linear differential operator, may be solved by Frobenius's series method (cf. Piaggio 1954, ch. 9). In this method, a solution is assumed of form

$$y = z^{s} \sum_{r=0}^{\infty} a_r z^r. \tag{2}$$

The exponent s is determined by substituting (2) into (1), equating the coefficients of lowest powers in z and solving the resulting quadratic (indicial) equation. Subsequent substitution of (2) into (1) then leads to a recurrence relation which when solved yields the coefficients a_r. The indicial equation normally gives two values for s, and in general it is therefore possible to obtain the complete primitive of (1). However, this approach leads only to expansions in increasing powers of z and it is to be expected that the results are generally not useful when $|z|$ is large. Expansions covering this case can be obtained in a formal way by 'inverting' the Frobenius method, i.e. by assuming solutions of form

$$y = z^{-s} \sum_{r=0}^{\infty} a_r z^{-r}. \tag{3}$$

The exponent s and the coefficients a_r are determined in a similar manner to that employed for the Frobenius method, except that here the indicial equation is obtained by equating the coefficients of highest powers in z . Before applying this method, it is of course necessary to remove any functions which cannot be expanded in inverse powers of z , such as $e^{\pm z}$.

To illustrate, this technique will now be applied to some well-known functions.

2.1 The general exponential integral function with argument large.

For $\mathrm{Re}(z) > 0$, the exponential integral function $Ei_p(z)$ is defined as (Placzek 1946, Busbridge 1950)

$$Ei_p(z) = \int_1^{\infty} e^{-uz} u^{-p} du, \tag{1}$$

and satisfies the differential equation

$$Ei_p''(z) + \left(1 - \frac{p-2}{z}\right) Ei_p'(z) - \frac{p-1}{z} Ei_p(z) = 0, \quad (2)$$

primes indicating differentiations with respect to z. To remove any exponential term, consider (2) when z is large. The dominant terms are $Ei_p''(z)$ and $Ei_p'(z)$; consequently the exponential behaviour is

$$Ei_p(z) \propto e^{-z}. \quad (3)$$

Therefore, a new function y(z) is defined by

$$Ei_p(z) = e^{-z} y(z), \quad (4)$$

which satisfies the equation

$$y'' - \left(1 + \frac{p-2}{z}\right) y' - \frac{y}{z} = 0. \quad (5)$$

In accordance with the above let

$$y = z^{-s} \sum_{r=0}^{\infty} a_r z^{-r}. \quad (6)$$

Substitution of (6) into (5) yields s=1 and

$$\frac{a_{r+1}}{a_r} = -\,(r+p-1)\,. \tag{7}$$

The relation $r! = r\,(r-1)!$ enables (7) to be solved for a_r and, choosing the constant of proportionality to achieve agreement with definition (1), we obtain the following familiar asymptotic expansion valid for large z and small p

$$Ei_p(z) = \frac{e^{-z}}{z}\left\{1 - \frac{p}{z} + \frac{p(p+1)}{z^2} + \cdots\right\}$$

$$= e^{-z}\sum_{r=0}^{\infty}\frac{(-1)^r\,(r+p-1)!}{(p-1)!\;z^{r+1}}\,. \tag{8}$$

Incidentally, this result could also have been derived by integrating (1) by parts.

2.2 <u>Modified Bessel functions with argument large</u>.

Modified Bessel functions are solutions of the equation

$$z^2 R_p''(z) + z R_p'(z) - (z^2+p^2)\,R_p(z) = 0\,. \tag{1}$$

The dominant terms for z large (but p not large) are $z^2 R_p''$ and $-z^2 R_p$, so that the exponential behaviour of R_p is given by $e^{\pm z}$. Therefore, two solutions are expected: one tending to zero as $z \to \infty$, the other increasing without bound in this limit. We also note from Watson (1944, § 7.23) that the modified Bessel functions $K_p(z)$ and $I_p(z)$ are defined by the relations

$$K_p(z) \sim \sqrt{\frac{\pi}{2z}}\, e^{-z} \left\{ 1 + O(z^{-1}) \right\}, z \to \infty, \quad (2)$$

$$I_p(z) \sim \frac{e^z}{\sqrt{2\pi z}} \left\{ 1 + O(z^{-1}) \right\}, z \to \infty. \quad (3)$$

We are thus led to write

$$K_p(z) \propto e^{-z}\, y(z), \quad (4)$$

$$I_p(z) \propto e^{z}\, y(-z). \quad (5)$$

Substitution of (4) into (1) yields for y(z)

$$z^2 y'' + (z - 2z^2) y' - (z + p^2) y = 0. \qquad (6)$$

If now

$$y = z^{-s} \sum_{r=0}^{\infty} a_r z^{-r}, \qquad (7)$$

is substituted into (6), we derive that $s=\frac{1}{2}$ and that a_r satisfies the recurrence relation

$$\frac{a_{r+1}}{a_r} = -\frac{(r+p+\frac{1}{2})(r-p+\frac{1}{2})}{2(r+1)}, \qquad (8)$$

a solution of which is

$$a_r = \frac{(r+p-\frac{1}{2})!\,(r-p-\frac{1}{2})!}{(-2)^r \quad r!}. \qquad (9)$$

By combining (7) with (4) and (5), and (2) and (3) therefore,

$$K_p(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z} \sum_{r=0}^{\infty} \frac{(r+p-\frac{1}{2})!\,(r-p-\frac{1}{2})!}{(p-\frac{1}{2})!\,(-p-\frac{1}{2})!\,r!} (-2z)^{-r}, \qquad (10)$$

and

$$I_p(z) = \frac{e^{z}}{\sqrt{2\pi z}} \sum_{r=0}^{\infty} \frac{(r+p-\frac{1}{2})!\,(r-p-\frac{1}{2})!}{(p-\frac{1}{2})!\,(-p-\frac{1}{2})!\,r!} (2z)^{-r}. \qquad (11)$$

2.3 <u>The Anger function with argument large</u>.

It is convenient to discuss only the deviation of the Anger function $\mathbf{J}_p(z)$ from the Bessel function $J_p(z)$. Let a function $j_p(z)$ therefore be defined by

$$j_p(z) = \frac{\pi}{\sin(\pi p)} \left\{ \mathbf{J}_p(z) - J_p(z) \right\}, \qquad (1)$$

satisfying the differential equation

$$z^2 j_p''(z) + z j_p'(z) + (z^2 - p^2) j_p(z) = z - p. \quad (2)$$

For large z, but small p, the equation for the dominant terms is

$$z^2 j_p \sim z \ , \text{ i.e. } \quad j_p(z) \sim \frac{1}{z} . \quad (3)$$

Although it is of lower order in z than $z^2 j_p$, the term on the right-hand side of (2) has to be retained, since otherwise we just end up with the Bessel function. An expansion for j_p in inverse powers of z is now obtained most easily by writing $j_p = \bar{j}_p + \bar{\bar{j}}_p$, where $\bar{j}_p$ and $\bar{\bar{j}}_p$ satisfy the equations

$$z^2 \bar{j}_p'' + z \bar{j}_p' + (z^2 - p^2) \bar{j}_p = z , \quad (4)$$

$$z^2 \bar{\bar{j}}_p'' + z \bar{\bar{j}}_p' + (z^2 - p^2) \bar{\bar{j}}_p = -p . \quad (5)$$

By considering dominant terms, it will be seen that for large z, $\bar{j}_p \sim 1/z$; and hence $a_1 = 1$ in

$$\bar{j}_p = \sum_{r=1}^{\infty} a_r z^{-r}. \tag{6}$$

Substitution of (6) into (4) leads to the recurrence relation

$$\frac{a_{r+2}}{a_r} = -(r-p)(r+p), \tag{7}$$

a solution of which is

$$a_r \propto (-1)^{\frac{1}{2}r} 2^r \left(\tfrac{1}{2}r-\tfrac{1}{2}p-1\right)!\left(\tfrac{1}{2}r+\tfrac{1}{2}p-1\right)! . \tag{8}$$

In view of (6) and the fact that $a_1 = 1$ therefore

$$\bar{j}_p(z) = \sum_{r=1,3,5,\cdots}^{\infty} (-4)^{\frac{1}{2}(r-1)} \frac{\left(\frac{1}{2}r-\frac{1}{2}p-1\right)!\left(\frac{1}{2}r+\frac{1}{2}p-1\right)!}{\left(\frac{1}{2}p-\frac{1}{2}\right)!\left(-\frac{1}{2}p-\frac{1}{2}\right)!\, z^r}. \tag{9}$$

We note that (5) is even in z and thus assume an expansion of the form

$$\bar{\bar{j}}_p(z) = -p \sum_{r=1}^{\infty} b_r z^{-2r}, \tag{10}$$

where $b_1 = 1$ and b_r satisfies the recurrence relation

$$\frac{b_{r+1}}{b_r} = -(4r^2 - p^2), \tag{11}$$

which has a solution

$$b_r = C\,(-4)^r (r - \tfrac{1}{2}p - 1)!\,(r + \tfrac{1}{2}p - 1)!\,, \tag{12}$$

where C is determined by the condition that $b_1 = 1$.

Adding the expansions for $\bar{J}_p$ and $\bar{\bar{J}}_p$, and combining the result with (1), we obtain the following expansion for the Anger function (cf. Watson 1944, p.315)

$$\mathbf{J}_p(z) = J_p(z) + \frac{\sin p\pi}{\pi z}\left[\left\{1 - \frac{1^2-p^2}{z^2} + \frac{(1^2-p^2)(3^2-p^2)}{z^4} - \cdots\right\}\right.$$

$$\left. - \frac{p}{z}\left\{1 - \frac{2^2-p^2}{z^2} + \frac{(2^2-p^2)(4^2-p^2)}{z^4} - \cdots\right\}\right.$$

$$= J_p(z) + \frac{\sin p\pi}{\pi}\left[\sum_{r=1,3,5,\cdots}^{\infty} (-1)^{\frac{1}{2}(r-1)} \frac{2^{r-1}(\frac{1}{2}r - \frac{1}{2}p - 1)!\,(\frac{1}{2}r + \frac{1}{2}p - 1)!}{(-\frac{1}{2}p - \frac{1}{2})!\,(\frac{1}{2}p - \frac{1}{2})!\; z^r}\right.$$

$$\left. - p\sum_{r=1,2,4,\cdots}^{\infty} (-4)^{r-1}\frac{(r - \frac{1}{2}p - 1)!\,(r + \frac{1}{2}p - 1)!}{(-\frac{1}{2}p)!\,(\frac{1}{2}p)!\; z^{2r}}\right]. \tag{13}$$

2.4 Confluent hypergeometric functions with argument large.

Confluent hypergeometric functions satisfy Kummer's equation

$$z F'' + (c - z) F' - a F = 0 . \tag{1}$$

To determine any exponential terms, consider the approximate equation for large z (but small a and c) :

$$z F'' - z F' \simeq 0 , \tag{2}$$

The general solution of (2) is

$$F \propto e^{z} + A , \tag{3}$$

and for the exponentially large solution, F_1 say, we write

$$F_1 \propto e^{z} y . \tag{4}$$

By (4) and (1), the equation for y is

$$z y'' + (c + z) y' + (c - a) y = 0 , \tag{5}$$

and writing

$$y = z^{-s_1} \sum_{r=0}^{\infty} a_r z^{-r}, \tag{6}$$

we derive that $s_1 = c - a$ and that a_r is a solution of the recurrence relation

$$\frac{a_{r+1}}{a_r} = \frac{(r-a+1)(r+c-a)}{(r+1)} . \tag{7}$$

Thus

$$a_r = \frac{(r-a)!\,(r+c-a-1)!}{r!} \tag{8}$$

It follows therefore by (4) that

$$F_1 = e^z z^{a-c} \sum_{r=0}^{\infty} \frac{(r-a)!\,(r+c-a-1)!}{(-a)!\,(c-a-1)!\,r!} z^{-r}, \tag{9}$$

where the proportionality constant has been chosen such that the coefficient of z^0 in the series is unity.

Let the second solution indicated by (3) be denoted by F_2 . Then if

$$F_2 \propto z^{-s_2} \sum_{r=0}^{\infty} b_r z^{-r} \tag{10}$$

is substituted into (1), it follows by earlier methods that $s_2 = a$ and that b_r is

$$b_r = (-1)^r \frac{(r+a-1)!\,(r+a-c)!}{r!} \ . \tag{11}$$

Thus

$$F_2 = z^{-a} \sum_{r=0}^{\infty} \frac{(r+a-1)!\,(r+a-c)!}{(a-1)!\,(a-c)!\,r!} (-z)^{-r} , \tag{12}$$

where again the proportionality constant makes the coefficient of z^0 in the series equal to unity.

F_1 and F_2 can now be identified in terms of commonly used confluent hypergeometric functions. We note from Slater (1960, § 4.1.1) that if $|z| \to \infty$, $\mathrm{Re}(z) > 0$,

$${}_1F_1(a, c, z) = \frac{(c-1)!}{(a-1)!} e^z z^{a-c} \left\{ 1 + O\left(|z|^{-1}\right) \right\}, \tag{13}$$

and if $|z| \to \infty$, $\mathrm{Re}(z) < 0$,

$$ {}_1F_1(a,c,z) = \frac{(c-1)!}{(c-a-1)!} (-z)^{-a} \left\{ 1 + O\left(|z|^{-1}\right) \right\}. \qquad (14) $$

It is evident therefore that for large $|z|$

$$ {}_1F_1(a,c,z) = \frac{(c-1)!}{(a-1)!} F_1 + \frac{(c-1)!}{(c-a-1)!} F_2 . \qquad (15) $$

Jeffreys (1962,p.97) denotes F_2 by U(a,c,z) and F_1 by V(a,c,z).

3. DERIVATION OF GREEN-TYPE, TRANSITIONAL AND UNIFORM ASYMPTOTIC EXPANSIONS.

3.1 Introduction.

Many linear second-order differential equations, notably those of the Sturm-Liouville type, can be written in the form

$$\frac{d^2 y}{dz^2} = \chi(a^2, z)\, y, \tag{1}$$

where a^2 is a parameter. As far as can be ascertained, the study of the asymptotic properties of the solutions of this type of equation was initiated by Liouville (1837) and Green (1837). They derived the following zero-order approximations $y^0_\pm$ to a pair of independent solutions of (1)

$$y^0_\pm = \chi^{-1/4} e^{\pm\int^z \chi^{1/2} dz} . \tag{2}$$

Henceforth, $y^0_\pm$ as given by (2) will be referred to as the Liouville - Green factors.+

Langer (1934b) considered an equation of form

$$\frac{d^2y}{dx^2} = \left\{ a^2 p(x) - q(x) \right\} y, \tag{3}$$

where a^2 is a large parameter, p is a continuous function of x, and q is an analytic function of x and bounded with respect to a^2. He showed by changing the dependent and independent variables that, if p has no zero in the x - interval, (a,b) say, (3) can be reduced to an equation of the form

$$\frac{d^2w}{dt^2} = \left\{ \rho^2 + \sigma(t) \right\} w, \tag{4}$$

and that, provided the integral

$$\int_a^b \left| \frac{q}{p^{1/2}} - \frac{5p'^2 - 4pp''}{16p^{5/2}} \right| dx \tag{5}$$

exists, a pair of independent solutions of (3) can

+ In physical investigations, the designation 'W.K.B.(J) approximation' is common in recognition of the application of (2) by Wentzel (1926), Kramers (1926) and Brillouin (1926) to eigenvalue problems in wave mechanics, and researches by Jeffreys (1924) on connexion formulae linking such approximations on opposite sides of any zero of χ .

indeed be represented asymptotically in the expected form

$$y_{\pm} = p^{-1/4} e^{\pm\int^{x} p^{1/2}\,dx} \sum_{r=0}^{n} \frac{\alpha_r(x)}{a^r} + R_{n+1}(x,a). \qquad (6)$$

Here, R_{n+1} is a remainder after n terms whose magnitude can in some particular cases be calculated by more precise analysis such as that given recently by Olver (1961). This result is not quite satisfactory in that it only shows that for each value of n, pairs of solutions exist which have the asymptotic form (6). Olver (1955a) + showed in fact that this dependence on n is unnecessarily restrictive, and that pairs of solutions exist given by (6) for all values of n. This means essentially that for large enough a^2, R_{n+1} tends uniformly to zero as $n \to \infty$, so that now n in (6) becomes an arbitrary, unbounded integer. Giving new proofs, Olver extended Langer's results to include the behaviour of the solutions for complex independent variables, and also those cases

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+ This paper, and also Jeffreys (1962), should be consulted for references to earlier work on the existence theory of asymptotic solutions in intervals free from turning points.

where p possesses a double pole in the interval of integration, q being allowed to have a simple or double pole at the same point. The coefficients α_r were shown to satisfy a recurrence relation of integro-differential form, a result which incidentally also follows from equation 4 on p.549 of Langer (1934b), for real variables at least.

In subsequent sections, we will adopt the more direct approach of deriving the exact integro-differential equation satisfied by the solutions of (1) from which the appropriate Liouville-Green factors have been removed, rather than just the recurrence relation referred to above. Thus it proves unnecessary to <u>assume</u> a definite form, such as (6), for the expansions of the solutions of the given differential equation, thereby obviating any lengthy discussions, otherwise necessary, to justify expressions of type (6). The manner in which this integro-differential equation is derived moreover suggests, in each case considered to date, a new independent variable reducing it to simple polynomial form. Once in this form it becomes a straightforward routine process to solve the equation directly by an iterative procedure yielding Green-type expansions, valid away from any transition point+.

If in (3) p possesses a simple zero in the interval of integration, then, away from this turning point, (3) has solutions that may be represented asymptotically by linear combinations of y_+ and y_- given by (6). However, solutions on opposite sides of the turning point cannot thereby be linked, since the expressions (6) diverge rapidly near the turning point. Indeed, there is a discontinuous change in the constants of the linear combinations on passing through the turning point (cf. Stokes 1864,1871,1902). Gans (1915) and Jeffreys (1924) formulated methods for deriving the appropriate linking constants. The failure of the Green-type expansions in the vicinity of turning points is due to the fact that one tries to approximate the solutions of the given equation by means of elementary functions which are not sufficiently flexible to describe their behaviour throughout these regions. If this restriction to elementary functions is dropped, it becomes possible to derive asymptotic expansions for the

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+ For the definition of a transition point as it is used here, see Olver (1955a).

solutions which remain uniformly valid throughout the domain of validity of the given equation, and the difficulties due to the Stokes phenomenon are obviated. This idea was originally taken up by Jeffreys (1924) who showed that a pair of solutions of (3) near a simple zero of ρ may be asymptotically represented by those of a related simpler equation, in this case Airy functions. Langer elaborated this procedure and proved results of less restricted applicability, showing first for real functions ρ (1931), and subsequently for complex variables (1932), that on transforming both the dependent and independent variables, (3) is reducible to an equation of form

$$\frac{d^2 w}{dz^2} = \left\{ \alpha^2 z + f(z) \right\} w, \tag{7}$$

whose solutions are asymptotic to Bessel functions of order one-third, the solutions of the related equation. Later (1949a,b) Langer obtained formal uniform expansions of the solutions of (7) involving a solution of the less recondite related equation

$$\frac{d^2 W}{dz^2} = \alpha^2 z W, \tag{8}$$

and its first derivative. These results apply only to real values of the independent variable however.

Giving different proofs Olver (1955a) derived solutions of (7) in a form similar to that given by Langer but extended their domain of validity to include complex values of the independent variable (see also Cherry (1950)). He showed that if $P_i(z)$ $(i = 1, 2, 3)$ are three functions, any pair of which are independent solutions of the related equation (8) that tend exponentially to zero as $|z| \to \infty$ in one of three sectors S_i, then, in each one of these sectors a domain D_i may be specified in which the solution to (3) is expressible in the form

$$y_i(z) = P_i(a^{2/3}z) \sum_{r=0}^{\infty} \frac{A_r(z)}{a^{2r}} + a^{-4/3} P_i'(a^{2/3}z) \sum_{r=0}^{\infty} \frac{B_r(z)}{a^{2r}}. \qquad (9)$$

The coefficients $A_r(z)$ and $B_r(z)$ are given by the relations

$$\left.\begin{aligned} A_0(z) &= 1, \\ A_{r+1}(z) &= -\tfrac{1}{2} B_r'(z) + \tfrac{1}{2}\int^{z} f(z) B_r(z)\, dz, \\ B_r(z) &= \tfrac{1}{2} z^{-\frac{1}{2}} \int_0^{z} z^{-\frac{1}{2}} \left\{ f(z) A_r(z) - A_r''(z) \right\} dz. \end{aligned}\right\} \qquad (10)$$

These double recurrence relations are by no means easy to solve analytically and, indeed, recourse has usually to be made to numerical methods (see, for instance, Olver 1955a, p.315; 1955b, pp.314 and 342).

The method introduced in this thesis consists essentially of the application of the Mellin transform to the integro-differential equation satisfied by $u \equiv y/y_+^0$ and $v \equiv y/y_-^0$, where y obeys an equation of type (1) and $y_\pm^0$ are the corresponding Liouville-Green factors given by (2). The difference equations that result are solved by successive substitutions, and uniform expansions are obtained for u and v as a series of Mellin-Barnes integrals. Explicitly the results are arranged as series in negative powers of the order; and also as a more complicated expression involving a modified Bessel function of order one-third and its first derivative. From either of these representations the usual Green-type expansions may be recovered; and in addition an expansion valid in the difficult transitional region is obtained.

The advantages of the present method over those of Langer and Olver are two-fold. Firstly, it avoids the occurrence of double recurrence relations such as (10), and thereby of processes of numerical differentiation and integration; and also of results derived from a knowledge of the integral representation of a solution of the given differential equation - e.g. Meissel's expansion (cf. Olver 1955b, p.342). Secondly, no a priori assumptions need be made about the form of the expansions.

In the preceding critical discussion, we have concentrated on the two cases covered by theorems A and B in Olver (1955a), namely those cases where the function in the leading term of (3) has i) no zero, and ii) a simple zero in the interval of integration. Similar considerations apply to more complicated cases such as are covered in, for instance, Olver (1956, 1958) and Thorne (1957).

3.2 The integro-differential equation.

We have seen in the foregoing section that the zero-order approximations to the differential equation

$$\frac{d^2 y}{dz^2} = \chi(z)\, y, \qquad (1)$$

are, for certain restrictions on γ,

$$y_{\pm}^{0} = \gamma^{-1/4} e^{\pm \int^{z} \gamma^{1/2} dz} . \tag{2}$$

To investigate the validity of this approximation, we reproduce an argument given by Olver (1961). Suppose that the differential equation satisfied by $y_{\pm}^{0}$ is

$$\frac{d^2 y}{dz^2} = \gamma(z)\{1 - f(z)\} y . \tag{3}$$

Substitution for $y_{\pm}^{0}$ yields

$$f(z) = \frac{4\gamma\gamma'' - 5\gamma'^2}{16\gamma^3} = -\frac{1}{\gamma^{3/4}} \frac{d^2}{dz^2}\left(\gamma^{-1/4}\right) . \tag{4}$$

Approximation (2) is therefore reasonable provided $|f(z)|$ is small throughout the integration interval. Equation (4) shows that this condition is met if $\gamma^{-1/4}$ is a bounded, slowly-varying function of z .

To derive later approximations, let a pair of independent solutions be written

$$\left.\begin{aligned} y_1 &= \chi^{-1/4}\, e^{+\int^x \chi^{1/2}\,dx}\, u, \\ y_2 &= \chi^{-1/4}\, e^{-\int^x \chi^{1/2}\,dx}\, v, \end{aligned}\right\} \qquad (5)$$

where apart from arbitrary multiplying constants u and v must be $O(1)$. The expression for u is obtainable from that for v simply by a reversal of sign of $\chi^{\frac{1}{2}}$, so that henceforth it is only necessary to consider v in detail. Substitution for y_2 in (1) leads to the following differential equation satisfied by v

$$v'' - \left(2\chi^{\frac{1}{2}} + \tfrac{1}{2}\chi_1\chi^{-1}\right)v' + \tfrac{1}{16}\chi^{-2}\left(5\chi_1^{\,2} - 4\chi\chi_2\right)v = 0, \qquad (6)$$

in which $d\chi/dx$ is denoted by χ_1 and $d^2\chi/dx^2$ by χ_2. Primes as superscripts on the dependent variable indicate differentiations with respect to the independent variable.

Since $-2\chi^{\frac{1}{2}}v'$ is the only term in (6) that increases with x, v' must be small $O(\chi^{-\frac{1}{2}})$ when $|\chi^{\frac{1}{2}}|$ is large. Thus V can be found by successive substitutions in the form

$$V = 1 + O\left(\gamma^{-\frac{1}{2}}\right), \tag{7}$$

from the following equation obtained from (6) by a rearrangement of the terms,

$$V' = \frac{1}{2}\gamma^{-\frac{1}{2}}V'' - \frac{1}{4}\gamma_1\gamma^{-\frac{3}{2}}V' + \frac{1}{32}\gamma^{-\frac{5}{2}}\left(5\gamma_1^2 - 4\gamma\gamma_2\right)V. \tag{8}$$

Integration throughout, followed by one integration by parts, yields

$$V = \frac{1}{2}\gamma^{-\frac{1}{2}}V' + \frac{1}{32}\int^x \gamma^{-\frac{5}{2}}\left(5\gamma_1^2 - 4\gamma\gamma_2\right)V\,dx. \tag{9}$$

If V_i is the i th order term in the expansion for V, the next higher order term V_{i+1} is given by

$$V_{i+1} = \frac{1}{2}\gamma^{-\frac{1}{2}}V_i' + \frac{1}{32}\int^x \gamma^{-\frac{5}{2}}\left(5\gamma_1^2 - 4\gamma\gamma_2\right)V_i\,dx. \tag{10}$$

Reversal of the sign of $\gamma^{\frac{1}{2}}$ shows that

$$u_i = (-1)^i \, v_i \, . \tag{11}$$

It may happen in a particular application that γ is separable into a large and a small part, K and ε, respectively. Then if

$$\gamma = K + \varepsilon \, , \tag{12}$$

a pair of independent solutions may be more conveniently written

$$y_1 = K^{-\frac{1}{4}} e^{\int^x K^{\frac{1}{2}} dx} u, \quad y_2 = K^{-\frac{1}{4}} e^{-\int^x K^{\frac{1}{2}} dx} v, \tag{13}$$

and the recurrence relation for v becomes

$$v_{i+1} = \frac{1}{2} K^{-\frac{1}{2}} v_i' + \frac{1}{32} \int^x K^{-\frac{5}{2}} \left(5K_1^2 - 4KK_2 - 16\varepsilon K^2 \right) v_i \, dx \tag{14}$$

with $K_1 = dK/dx$, $K_2 = d^2K/dx^2$. Equation (11) remains unchanged.

In all applications so far it has been found possible to reduce (10) and (14) to simple polynomial-like form by the introduction of a suitable new variable, q, say, which in its most general form may be written

$$q = \int^{x} \left(A\gamma^{\frac{1}{2}} + B\gamma^{-\frac{3}{2}}\gamma_2 \right) dx \, ; \tag{15}$$

A and B are parameters independent of x, but may involve the order.

It will be noted that in (10) and (14) the lower limit of integration has been left unspecified --- in each particular case this limit can be chosen such that the simplest possible expansions for v and u result.

4. ASYMPTOTIC EXPANSIONS FOR MODIFIED BESSEL FUNCTIONS OF LARGE ORDER.

4.1 The integro-differential equation.

The modified Bessel functions are solutions of the differential equation (cf. (2.2.1))

$$z^2 R_p''(z) + z R_p'(z) - (z^2 + p^2) R_p(z) = 0 . \tag{1}$$

This equation has transition points at $z = \pm i p$, and it will be assumed in this section and the next that $|\arg z/p| < \frac{1}{2}\pi$. In agreement with customary practice, the exponentially decreasing solution is taken to be $K_p(z) \sim (\frac{1}{2}\pi)^{\frac{1}{2}} z^{-\frac{1}{2}} e^{-z}$ (see, for example, Watson 1944, p.202; Erdélyi et al. 1953, § 7.4.1). The exponentially increasing solution is defined by $\mathfrak{R}_p(z) \sim (\frac{1}{2}\pi)^{\frac{1}{2}} z^{-\frac{1}{2}} e^{z}$, as introduced by Dingle (1959, pp. 280-281). This latter choice is particularly convenient in that $\mathfrak{R}_p(z)$ is real when both p and z are real, and also possesses a simple asymptotic expansion which does not contain any exponentially small terms.

Equation (1) is first brought into the standard form (3.2.1) by a transformation of independent variable suggested by Langer (1931, p.52; 1937, pp. 673-675), namely

$$z = e^{x}. \tag{2}$$

In terms of this variable it follows that

$$\frac{d^2}{dx^2} R_p(e^x) = (e^{2x}+p^2)\, R_p(e^x) ; \tag{3}$$

thus

$$\left.\begin{aligned} \chi &= e^{2x}+p^2, \\ \chi_1 &= 2e^{2x} = 2z^2, \\ \chi_2 &= 4e^{2x} = 4z^2. \end{aligned}\right\} \tag{4}$$

The criterion for the validity of (3.2.2) is certainly satisfied in the region of interest, so that the results of the previous section may be applied. In accordance with relations (3.2.5), we write, in terms of the original variable z ,

$$y_1 = (z^2+p^2)^{-\frac{1}{4}} \exp\left\{ -p \sinh^{-1}\left(\frac{p}{z}\right) + (z^2+p^2)^{\frac{1}{2}} \right\} u, \qquad (5)$$

$$y_2 = (z^2+p^2)^{-\frac{1}{4}} \exp\left\{ p \sinh^{-1}\left(\frac{p}{z}\right) - (z^2+p^2)^{\frac{1}{2}} \right\} v. \qquad (6)$$

Considering the asymptotic behaviour of (5) and (6) for $z \to \infty$, it is evident that y_1 can be identified with $\mathcal{I}_p$, and y_2 with K_p .

Postponing for the moment the determination of the constants of proportionality, we find from (3.2.9) and (4) that

$$v = \frac{z}{2(z^2+p^2)^{\frac{1}{2}}} v' + \frac{1}{8} \int^{z} \left\{ \frac{5z^3}{(z^2+p^2)^{\frac{5}{2}}} - \frac{4z}{(z^2+p^2)^{\frac{3}{2}}} \right\} v \, dz. \qquad (7)$$

A variable reducing this equation to polynomial-like form follows from (3.2.15) with $A = 0$, $B = -\frac{1}{4}p$; i.e. we put

$$q = \frac{\rho}{(z^2+\rho^2)^{\frac{1}{2}}} \qquad (8)^+$$

Integro-differential equation (7) then reduces to

$$2pV = q^2(q^2-1)V' + \frac{1}{4}\int^q (5q^2-1)\,V\,dq\,. \qquad (9)$$

4.2 <u>Green-type expansions.</u>

The recursive relation (3.2.10) becomes

$$2pV_{i+1} = q^2(q^2-1)V_i' + \frac{1}{4}\int_0^q (5q^2-1)\,V_i\,dq\,. \qquad (1)$$

(To get each contribution to V in the simplest form, the lower limit of integration in (1) has been taken

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+ This variable, incidentally, was also used by Olver (1955b, p.333) equal to zero.) Starting from $V_0 = 1$, (1) yields successively

$$V_1 = \frac{q}{24p}\,(5q^2 - 3),$$

$$V_2 = \frac{q^2}{1152\,p^2}\,(385\,q^4 - 462\,q^2 + 81),$$

$$V_3 = \frac{q^3}{414720\;p^3}\,(425\,425\,q^6 - 765\,765\,q^4$$

$$+ 369\,603\,q^2 - 30\,375),$$

$$V_4 = \frac{q^4}{39\,813\,120\,p^4}\,(185\,910\,725\,q^8 - 446\,185\,740\,q^6$$

$$+ 349\,922\,430\,q^4 - 94\,121\,676\,q^2 + 4\,465\,125),$$

$$V_5 = \frac{q^5}{6\,688\,604\,160\,p^5}\,(188\,699\,385\,875\,q^{10}$$

$$- 566\,098\,157\,625\,q^8 + 614\,135\,872\,350\,q^6$$

$$- 284\,499\,769\,554\, q^4 + 49\,286\,948\,607\, q^2$$

$$- 1\,519\,035\,525\big). \qquad (2)$$

The terms in (2) are found to agree with those calculated by W.G. Bickley in the British Association Tables (1960, p.xxxv). In chapter 10 we derive analytic expressions from which may be deduced the first five terms in highest powers and the first three in lowest powers of q for any v_i .

From its definition and (4.1.6), the full Green-type expansion for $K_p(z)$ becomes

$$K_p(z) = \left(\frac{\pi q}{2p}\right)^{\frac{1}{2}} \exp\left\{ p \sinh^{-1}\left(\frac{p}{z}\right) - \left(z^2 + p^2\right)^{\frac{1}{2}} \right\} \sum_{i=0}^{\infty} v_i . \quad (3)$$

Similarly, by (4.1.5) and (3.2.11),

$$\mathcal{K}_p(z) = \left(\frac{\pi q}{2p}\right)^{\frac{1}{2}} \exp\left\{ \left(z^2 + p^2\right)^{\frac{1}{2}} - p \sinh^{-1}\left(\frac{p}{z}\right) \right\} \sum_{i=0}^{\infty} (-1)^i v_i .$$

$$(4)$$

The results obtained in this section are applicable for

$|\arg z/p| < \frac{1}{2}\pi$; their validity may be extended to other phase ranges of z/p (but excluding the imaginary axis) with the use of the continuation formulae (cf. Watson 1944, p.80; Dingle 1959a, p.280)

$$K_p(z e^{m\pi i}) = e^{-mp\pi i} K_p(z) - \pi i \frac{\sin mp\pi}{\sin p\pi} I_p(z), \quad (5)$$

$$\mathcal{K}_p(z) = \tfrac{1}{2}\pi \left\{ I_p(z) + I_{-p}(z) \right\}, \quad (6)$$

$$I_{\pm p}(z e^{m\pi i}) = e^{\pm mp\pi i} I_{\pm p}(z), \quad (7)$$

with the functions $I_{\pm p}$ defined in terms of Bessel functions of the first kind $J_{\pm p}$ by

$$I_{\pm p}(z) = e^{\mp \frac{1}{2} p\pi i} J_{\pm p}(iz). \quad (8)$$

4.3 <u>The Mellin transform of v</u>.

The Green-type expansions derived so far break down completely in the transitional region, $|\arg z/p| \simeq \frac{1}{2}\pi$ in the case of modified Bessel functions. This is simply because χ in (3.2.1) becomes small and $\chi^{-\frac{1}{4}}$ is no longer slowly varying, causing the series for v and u to diverge rapidly. If solutions are to be obtained that are valid in the neighbourhood of transition points, it becomes necessary to use more powerful methods for solving the integro-differential equation (3.2.9) than the method described in the preceding two sections. In the present section, this equation is solved with the aid of the Mellin transform (cf. Morse and Feshbach 1953, pp. 469-471; Titchmarsh 1948, §§ 1.29 and 7.7; Doetsch 1950, chapter 11, 2, and 1955, chapters 5 and 6).

To illustrate the method for the particular case of modified Bessel functions, the exact integro-differential equation to be solved is, by (4.1.9),

$$2pv = q^2(q^2-1)v' + \frac{1}{4}\int_0^q (5q^2-1)\,v\,dq. \qquad (1)$$

For large q, the first approximation to v is given by the solution of

$$2pV = q^4 V' + \frac{5}{4}\int_0^q q^2 V\,dq. \qquad (2)$$

If a new variable

$$x = -\frac{3q^3}{2p} \qquad (3)$$

is introduced and V written as a Mellin-Barnes integral, thus

$$V(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M_0(\mu)\, x^{-\mu}\, d\mu, \tag{4}$$

c being a constant whose range of values will be determined when explicit expansions are considered, then individual terms in (2) may be written

$$\begin{aligned} q^4 V' &= \frac{\rho}{\pi i}\int_{c-i\infty}^{c+i\infty} \mu M_0(\mu)\, x^{-(\mu-1)}\, d\mu \\ &= \frac{\rho}{\pi i}\int_{c-i\infty}^{c+i\infty} (\mu+1) M_0(\mu+1)\, x^{-\mu}\, d\mu, \end{aligned} \tag{5}$$

provided there are no poles in the strip $c-1 < R_e(\mu) < c$;

and also

$$\begin{aligned} \frac{5}{4}\int_0^q q^2 V\, dq &= \frac{5\rho}{36\pi i}\int_{c-i\infty}^{c+i\infty} \frac{M_0(\mu)}{\mu-1}\, x^{-(\mu-1)}\, d\mu \\ &= \frac{5\rho}{36\pi i}\int_{c-i\infty}^{c+i\infty} \frac{M_0(\mu+1)}{\mu}\, x^{-\mu}\, d\mu, \end{aligned} \tag{6}$$

with the same proviso. Equation (2), in combination with (5) and (6), yields therefore

$$\int_{c-i\infty}^{c+i\infty} (1/\mu)\left\{(\mu+\tfrac{1}{6})(\mu+\tfrac{5}{6})M_0(\mu+1) - \mu M_0(\mu)\right\} x^{-\mu}\, d\mu = 0. \tag{7}$$

If (7) is to be true for all values of x, the integrand must be zero for all μ. Hence $M_0(\mu)$ must obey the difference equation

$$\frac{1}{\mu}\left(\mu+\frac{1}{6}\right)\left(\mu+\frac{5}{6}\right)M_0(\mu+1) - M_0(\mu) = 0, \tag{8}$$

which is satisfied by

$$M_0(\mu) = (\mu-1)!\left(-\mu-\frac{5}{6}\right)!\left(-\mu-\frac{1}{6}\right)!. \tag{9}$$

According to (4) a solution of (2) is therefore

$$V(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(\mu-1)!\left(-\mu-\frac{5}{6}\right)!\left(-\mu-\frac{1}{6}\right)!\;x^{-\mu}\,d\mu. \tag{10}$$

For future reference it is noted that $V(x)$ is just the Mellin-Barnes integral representation of the function (see, for instance, Erdélyi et al. 1953, vol. I, p.275)

$$\left(\frac{4\pi}{3x}\right)^{\frac{1}{2}} e^{1/2x} K_{\frac{1}{3}}\left(\frac{1}{2x}\right) = \frac{\pi^{\frac{3}{2}}\,2^{\frac{5}{3}}}{x^{\frac{1}{6}}\,3^{\frac{1}{3}}}\,e^{1/2x}\,Ai\left\{\left(\frac{3}{4x}\right)^{\frac{2}{3}}\right\}, \tag{11}$$

where the Airy function $Ai(x)$ is as defined by J.C.P. Miller

(British Association 1946).

Later approximations to V are found by writing

$$V = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M(\mu)\, x^{-\mu}\, d\mu , \tag{12}$$

and substituting in (1). The condition that the resulting generalization of (7) is zero for all x leads to the difference equation

$$\left.\begin{aligned} &\left(\mu+\tfrac{1}{6}\right)\left(\mu+\tfrac{5}{6}\right) M(\mu+1) - \mu M(\mu) \\ &\qquad = \left(-\tfrac{3}{2p}\right)^{\frac{2}{3}}\left(\mu+\tfrac{1}{6}\right)^{2} M\left(\mu+\tfrac{1}{3}\right), \end{aligned}\right\} \tag{13}$$

which, since p is large, is in a form suitable for solution by successive substitutions. Before proceeding to the solution, however, it is convenient to remove the zero-order approximation $M_0(\mu)$ first, by defining a function $m(\mu)$ such that

$$M(\mu) = (\mu-1)!\left(-\mu-\tfrac{5}{6}\right)!\left(-\mu-\tfrac{1}{6}\right)!\; m(\mu). \tag{14}$$

In terms of $m(\mu)$, (13) may now be written

$$m(\mu) = -\left(-\frac{3}{2p}\right)^{\frac{2}{3}} \mathop{S} \frac{\left(\mu+\frac{1}{6}\right)\left(\mu-\frac{2}{3}\right)!\left(-\mu-\frac{1}{2}\right)!}{\mu!\ \left(-\mu-\frac{5}{6}\right)!}\, m\left(\mu+\frac{1}{3}\right), \quad (15)$$

where $\mathop{S}$ is the summation operator as defined in, for instance, Nörlund (1923, pp.40-47) and Milne-Thomson (1933, § 8.0).

If m_i denotes the i th order term in the expression for m , the next higher term m_{i+1} is given by

$$m_{i+1}(\mu) = -\left(-\frac{3}{2p}\right)^{\frac{2}{3}} \mathop{S} \frac{\left(\mu+\frac{1}{6}\right)\left(\mu-\frac{2}{3}\right)!\left(-\mu-\frac{1}{2}\right)!}{\mu!\ \left(-\mu-\frac{5}{6}\right)!}\, m_i\left(\mu+\frac{1}{3}\right). \quad (16)$$

The initial constant term $m_0(\mu)$ will be taken equal to $\left[\left(-\frac{1}{6}\right)!\left(-\frac{5}{6}\right)!\right]^{-1} = (2\pi)^{-1}$, since this choice leads to agreement with the definition of V adopted earlier (e.g. (3.2.5), (4.2.2)). Equation (16) then yields

$$m_1(\mu) = -\left(-\frac{3}{2p}\right)^{\frac{2}{3}} \frac{1}{2\pi} \sum \frac{(\mu+\frac{1}{6})(\mu-\frac{2}{3})!\,(-\mu-\frac{1}{2})!}{\mu!\,(-\mu-\frac{5}{6})!} \,. \tag{17}$$

The result of this summation is expected to be of the form

$$C\ \frac{(\mu-\frac{2}{3})!\,(-\mu+\frac{1}{2})!}{(\mu-1)!\ (-\mu-\frac{5}{6})!}\ ;$$

differencing gives $C = -\frac{3}{5}$. Hence

$$m_1(\mu) = \frac{3}{10\pi}\left(-\frac{3}{2p}\right)^{\frac{2}{3}} \frac{(\mu-\frac{2}{3})!\,(-\mu+\frac{1}{2})!}{(\mu-1)!\ (-\mu-\frac{5}{6})!} \,. \tag{18}$$

In a similar manner it is found that

$$m_2(\mu) = -\frac{9}{100\pi}\left(-\frac{3}{2p}\right)^{\frac{4}{3}}\left(\mu-\frac{47}{42}\right) \frac{(\mu-\frac{1}{3})!\,(-\mu+\frac{1}{2})!}{(\mu-1)!\ (-\mu-\frac{7}{6})!} \,. \tag{19}$$

Next,

$$m_3(\mu) = \frac{9}{100\pi}\left(-\frac{3}{2p}\right)^2 S\left(\mu^2 - \frac{1}{36}\right)\left(\mu + \frac{1}{2}\right)\left(\mu - \frac{11}{14}\right). \quad (20)$$

This summation can be carried through by writing

$$S\left(\mu^2 - \frac{1}{36}\right)\left(\mu + \frac{1}{2}\right)\left(\mu - \frac{11}{14}\right) = \sum_{r=0}^{5} a_r \mu^r, \quad (21)$$

so that

$$\left(\mu^2 - \frac{1}{36}\right)\left(\mu + \frac{1}{2}\right)\left(\mu - \frac{11}{14}\right) = \Delta \sum_{r=0}^{5} a_r \mu^r,$$

where Δ is the differencing operator. The coefficients a_r may be determined by equating the coefficients of equal powers in μ. Alternatively, use may be made of the relation

$$S(\mu^r) = \frac{B_{r+1}(\mu)}{r+1},$$

where $B_r(\mu)$ are Bernoulli polynomials (cf. Nörlund 1923, p.18). By either route, it follows that

$$m_3(\mu) = \frac{9}{500\pi}\left(-\frac{3}{2p}\right)^{2} \mu\left(\mu^2-\frac{1}{4}\right)\left(\mu^2-\frac{20}{7}\mu+\frac{1459}{756}\right). \quad (22)$$

The determination of later approximations involves no further new techniques. The first five contributions are calculated to be

$$\begin{aligned} 2\pi\, M(\mu) = {} & (\mu-1)!\left(-\mu-\tfrac{5}{6}\right)!\left(-\mu-\tfrac{1}{6}\right)! \\ & + \frac{3}{5}\left(-\frac{3}{2p}\right)^{\frac{2}{3}}\left(\mu-\tfrac{2}{3}\right)!\left(-\mu-\tfrac{1}{6}\right)!\left(-\mu+\tfrac{1}{2}\right)! \\ & + \frac{9}{50}\left(-\frac{3}{2p}\right)^{\frac{4}{3}}\left(\mu+\tfrac{1}{6}\right)\left(\mu-\tfrac{47}{32}\right)\left(\mu-\tfrac{1}{3}\right)!\left(-\mu-\tfrac{5}{6}\right)!\left(-\mu+\tfrac{1}{2}\right)! \\ & + \frac{9}{250}\left(-\frac{3}{2p}\right)^{2}\left(\mu^2-\tfrac{1}{4}\right)\left(\mu^2-\tfrac{20}{7}\mu+\tfrac{1459}{756}\right) \\ & \qquad \times \mu!\left(-\mu-\tfrac{5}{6}\right)!\left(-\mu-\tfrac{1}{6}\right)! \end{aligned}$$

$$+\frac{27}{5000}\left(-\frac{3}{2p}\right)^{\frac{8}{3}}\left(\mu+\frac{5}{6}\right)\left(\mu^3-\frac{73}{14}\mu^2+\frac{45613}{5292}\mu-\frac{1567331}{349272}\right)$$

$$\times \left(\mu+\tfrac{1}{3}\right)!\left(-\mu-\tfrac{1}{6}\right)!\left(-\mu+\tfrac{1}{2}\right)!$$

$$+ \dots . \tag{23}$$

4.4 Green-type and transitional expansions.

From (4.3.12) and (4.3.23) it is now possible to derive expansions for Ψ in both ascending and descending powers of η. To achieve this, it is necessary to choose the separation constant c in (4.3.12) in such a way that the contour of integration separates the poles contributing to the ascending from those that give rise to the descending expansion. For the integral involving the zero-order term in (4.3.23) this means that $0 < c < \frac{1}{6}$; the first-order term dictates that c must satisfy the inequality $-\frac{1}{3} < c < \frac{5}{6}$, and so on (see also Dingle 1955, p.405).

4.41 Ascending expansion (Green-type)

With c chosen appropriately and the contour of integration closed such that it encloses only the poles lying to the left of the point where it crosses the real axis, evaluation of the residues at each contributing pole yields the following expansion for V (valid away from the transition point):

$$2\pi V = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(n-\tfrac{1}{6}\right)! \left(n-\tfrac{5}{6}\right)! \left(-\frac{3q^3}{2p}\right)^n$$

$$+ \frac{3}{5}\left(-\frac{3}{2p}\right)^{\frac{2}{3}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(n+\tfrac{5}{6}\right)!\left(n+\tfrac{1}{6}\right)! \left(-\frac{3q^3}{2p}\right)^{n+\frac{1}{3}}$$

$$+ \frac{9}{50}\left(-\frac{3}{2p}\right)^{\frac{4}{3}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(n+\tfrac{1}{2}\right)\left(n+\tfrac{25}{14}\right)$$

$$\times \left(n-\tfrac{1}{6}\right)!\left(n+\tfrac{7}{6}\right)! \left(-\frac{3q^3}{2p}\right)^{n+\frac{2}{3}}$$

$$+ \cdots \tag{1}$$

On ordering the terms, we obtain

$$V = 1 + \frac{q}{24p}\left(5q^2-3\right) + \frac{q^2}{1152p^2}\left(385q^4-462q^2+81\right)+\cdots, \tag{2}$$

which is just the Green-type expansion obtained earlier (see (4.2.2)).

4.42 Descending expansion (transitional)

If the contour of integration is closed to encompass only the poles lying to the right of the point at which it crosses the real axis, the expansion for V, valid in the transitional region, becomes

$$V(p,q) = \left(-\tfrac{2}{3}\right)!\left(\tfrac{1}{6}p\right)^{\frac{1}{6}} e^{-\frac{1}{6}\pi i} \cdot (\pi q)^{-\frac{1}{2}} \, .$$

$$\left[\left\{ 1 + \frac{\left(-\frac{1}{6}\right)!}{70\sqrt{\pi}}\, 3^{\frac{1}{3}} e^{\frac{1}{3}\pi i} p^{-\frac{4}{3}} - \frac{1}{225} p^{-2} + \cdots \right\} \right.$$

$$- \frac{\left(-\frac{1}{6}\right)!}{2q^2\sqrt{\pi}}\, 3^{\frac{1}{3}} e^{-\frac{2}{3}\pi i} p^{\frac{2}{3}} \left\{ 1 - \frac{\left(\frac{1}{6}\right)!}{5\sqrt{\pi}}\, 3^{\frac{2}{3}} e^{\frac{2}{3}\pi i} p^{-\frac{2}{3}} \right.$$

$$\left. + \frac{23}{3150} p^{-2} - \frac{947\left(\frac{1}{6}\right)!}{346\,500\sqrt{\pi}}\, 3^{\frac{2}{3}} e^{-\frac{1}{3}\pi i} p^{-\frac{8}{3}} + \cdots \right\}$$

$$- \frac{p}{3q^3} \left\{ 1 + \frac{\left(-\frac{1}{6}\right)!}{70\sqrt{\pi}}\, 3^{\frac{1}{3}} e^{\frac{1}{3}\pi i} p^{-\frac{4}{3}} - \frac{1}{225} p^{-2} + \cdots \right\}$$

$$- \frac{(-\frac{1}{6})!}{4q^4 \sqrt{\pi}} \, 3^{\frac{1}{3}} e^{-\frac{2}{3}\pi i} \, p^{\frac{2}{3}} \left\{ 1 - \frac{(\frac{1}{6})!}{5\sqrt{\pi}} 3^{\frac{2}{3}} e^{\frac{2}{3}\pi i} p^{-\frac{2}{3}} + \frac{23}{3150} p^{-2} + \cdots \right\}$$

$$+ \cdots \Big] . \tag{1}$$

The complete expression for $K_p(z)$ includes the Liouville-Green factor as given in (4.2.3) so that, in the region $|\arg z/p| \underline{\Omega} \frac{1}{2}\pi$, $K_p(z)$ is asymptotically represented by

$$K_p(z) = \left(\frac{\pi q}{2p}\right)^{\frac{1}{2}} \exp\left\{ p \sinh^{-1}\left(\frac{p}{z}\right) - (z^2+p^2)^{\frac{1}{2}} \right\} V(p,q) . \tag{2}$$

4.43 Limit $z \to \pm ip \; (q \to \infty)$.

Expansions for the values of $K_p(z)$ at the transition points $z = \pm ip$ are derived by combining (4.42.1) and (4.42.2) and proceeding to the limit $q \to \infty$. This yields

$$K_p(\pm ip) = 2^{-\frac{2}{3}}\left(-\tfrac{2}{3}\right)!\, e^{-\left(\pm p+\frac{1}{3}\right)\frac{1}{2}\pi i}\left(3p^2\right)^{-\frac{1}{6}}$$

$$\left\{1 + \frac{\left(-\frac{1}{6}\right)!}{70\sqrt{\pi}}\, 3^{\frac{1}{3}} e^{\frac{1}{3}\pi i} p^{-\frac{4}{3}} - \frac{1}{225} p^{-2} + \cdots\right\}. \qquad (1)$$

Corresponding expressions for $\mathcal{K}_p(z)$ may be deduced by noting that the relation $u_i = (-1)^i\, v_i$ in the integro-differential equation is equivalent to the transformation $p \rightarrow -p$ in the equation for v_i. Thus,

$$\mathcal{K}_p(z) = \left(\frac{\pi q}{2p}\right)^{\frac{1}{2}} \exp\left\{\left(z^2+p^2\right)^{\frac{1}{2}} - p \sinh^{-1}\left(\frac{p}{z}\right)\right\} v(-p, q). \qquad (2)$$

4.5 Uniform expansions involving modified Bessel functions of order one-third.

Since $K_{\frac{1}{3}}$ satisfies a second-order differential equation, it can reasonably be expected from the remarks following (4.3.10) that later contributions to v are expressible in terms of this function and its first derivative. However, the method of Mellin transforms introduced here avoids such a prior assumption, and also steers clear of the awkward double recurrence relations that appear in earlier treatments.

We now denote the function defined by (4.3.10) and (4.3.11) by $k(x)$, and take over the definition of $M(\mu)$ in terms of ν given by (4.3.12). The problem is to express each of the terms in (4.3.12) arising from $M(\mu)$ as some combination of k and its derivative, k'. This is straightforward for the first term; it is simply equal to k itself. The second term, or first-order contribution, is from (4.3.23)

$$\frac{3}{5}\left(-\frac{3}{2p}\right)^{\frac{2}{3}}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\mu-\tfrac{2}{3}\right)!\left(-\mu-\tfrac{1}{6}\right)!\left(-\mu+\tfrac{1}{2}\right)!\;x^{-\mu}\,d\mu. \quad (1)$$

One differentiation of (4.3.10) with respect to x and a subsequent real displacement in μ yields

$$k'(x) = \frac{-x^{-\frac{1}{3}}}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\mu-\tfrac{2}{3}\right)!\left(-\mu+\tfrac{1}{2}\right)!\left(-\mu-\tfrac{1}{6}\right)!\;x^{-\mu}\,d\mu, \quad (2)$$

so that (1) reduces to

$$-\frac{3}{5}\,x^{\frac{1}{3}}\left(-\frac{3}{2p}\right)^{\frac{2}{3}}k'(x). \quad (3)$$

Apart from a multiplying constant, the second-order contribution in (4.3.23) is

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\mu+\tfrac{1}{6}\right)\left(\mu-\tfrac{47}{42}\right)\left(\mu-\tfrac{1}{3}\right)!\left(-\mu-\tfrac{5}{6}\right)!\left(-\mu+\tfrac{1}{2}\right)!\,x^{-\mu}\,d\mu. \quad (4)$$

If (4) is arranged such that the difference between the arguments of the last two factorials is in agreement with the corresponding difference in (4.3.10), i.e. arranged to equal $\frac{2}{3}$, (4) becomes

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\mu+\tfrac{1}{6}\right)\left(\mu-\tfrac{47}{42}\right)\left(\mu^2-\tfrac{1}{4}\right)\left(\mu-\tfrac{1}{3}\right)!\left(-\mu-\tfrac{5}{6}\right)!\left(-\mu-\tfrac{3}{2}\right)!\,x^{-\mu}\,d\mu. \quad (5)$$

It is at this stage convenient to introduce an operator $\mathfrak{H}$, defined by its operation on a general function $P(\mu)$, as follows,

$$\mathfrak{H}\,P(\mu)\equiv\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\mu-\tfrac{1}{3}\right)!\left(-\mu-\tfrac{5}{6}\right)!\left(-\mu-\tfrac{3}{2}\right)!\,P(\mu)\,x^{-\mu}\,d\mu. \quad (6)$$

In this notation the following relations may be written down with the aid of equation (4.3.10)

$$\left.\begin{aligned}
x^{\frac{2}{3}}\, k(x) &= \mathfrak{A}\, 1, \\
x^{\frac{5}{3}}\, k'(x) &= -\,\mathfrak{A}\,(\mu+\tfrac{2}{3}), \\
x^{\frac{8}{3}}\, k''(x) &= \mathfrak{A}\,(\mu+\tfrac{2}{3})(\mu+\tfrac{5}{3}), \\
x^{\frac{11}{3}}\, k'''(x) &= -\,\mathfrak{A}\,(\mu+\tfrac{2}{3})(\mu+\tfrac{5}{3})(\mu+\tfrac{8}{3}), \\
x^{\frac{14}{3}}\, k^{iv}(x) &= \mathfrak{A}\,(\mu+\tfrac{2}{3})(\mu+\tfrac{5}{3})(\mu+\tfrac{8}{3})(\mu+\tfrac{11}{3}).
\end{aligned}\right\} \qquad (7)$$

The factorial functions in (5) are now the same as those in (7); it is thus a simple matter to express the former as linear combinations of the right-hand side of (7), and hence in terms of $k(x)$ and its derivatives, by solving the equation

$$a\, x^{\frac{14}{3}} k^{iv}(x) + b\, x^{\frac{11}{3}} k'''(x) + c\, x^{\frac{8}{3}} k''(x) + d\, x^{\frac{5}{3}} k'(x)$$

$$+\, e\, x^{\frac{2}{3}} k(x) = \mathfrak{A}\,(\mu+\tfrac{1}{6})(\mu+\tfrac{47}{42})(\mu^2-\tfrac{1}{4}), \qquad (8)$$

for the constants a, b, c, d and e. Equating coefficients in (8) yields

$$a = 1, \quad b = \frac{202}{21}, \quad c = \frac{2771}{126}, \quad d = \frac{187}{18}, \quad e = \frac{25}{144}. \tag{9}$$

Since (4.3.10) and (2) define k and k' unambiguously, relations (7) contain sufficient information to enable one to express the higher derivatives of k in terms of these two functions. It is, nevertheless, better to use the differential equation satisfied by k,

$$x^2 k'' + (2x+1) k' + \frac{5}{36} k = 0, \tag{10}$$

later derivatives being found by successive differentiations. Carrying out the necessary calculations, the following expression is obtained for the second-order contribution to V,

$$V_2 = \frac{-1}{12600}\left(-\frac{3}{2p}\right)^{\frac{4}{3}}\Big\{k(x)\left(40x^2 + 120x + 315\right) + k'(x)\left(240x^3 + 5400x + 2268\right)\Big\}. \tag{11}$$

Similarly,

$$v_3 = \frac{-1}{756000}\left(-\frac{3}{2p}\right)^2 \left\{ k(x)\left(2480x^3 + 3680x^2 + 19440x + 3780\right) + k'(x)\left(19015x^4 + 4320x^3 + 248220x^2 + 194400x + 27216\right)\right\}. \qquad (12)$$

Including the fourth-order term, simplifying with the relation $x = -3q^3/2p$, and arranging the result as two series multiplying k and k', respectively, the expansion for v may be written

$$2\pi v = k(x)\left\{1 + \frac{q}{p}\sum_{n=1}^{\infty}(-p)^n a_n(x)\, q^{-5n}\right\} + \frac{9}{10}\frac{q}{p}\, k'(x)\sum_{n=0}^{\infty}(-p)^n b_n(x)\, q^{-5n}, \qquad (13)^+$$

+ This expansion may be compared with the uniform expansions for Hankel functions in terms of Airy functions derived by Olver (1955b).

where the first few coefficients are

$$\left.\begin{aligned}
a_1(x) &= \frac{1}{2520}\left(8x^2+44x+63\right),\\
a_2(x) &= \frac{1}{56700}\left(124x^3+184x^2+972x+189\right),\\
a_3(x) &= \frac{1}{1571724000}\left(813\,680x^4+2330\,928x^3\right.\\
&\qquad +16415\,883x^2+6436\,584x\\
&\qquad \left.+523\,908\right),
\end{aligned}\right\}\quad (14a)$$

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$$b_0(x) = 1,$$

$$b_1(x) = \frac{1}{945}\left(20x^3+450x+189\right),$$

$$b_2(x) = \frac{1}{255\,150}\left(1480x^4+1080x^3+62055x^2\right.$$

$$\left.+48600x+6804\right),$$

$$b_3(x) = \frac{1}{98232750}\left(162260x^5 + 311960x^4 + 12938940x^3 + 14084235x^2 + 3742200x + 261954\right). \qquad (14b)$$

The Green-type and transitional expansions --- (4.2.3) and (4.42.2), respectively --- are recovered if in (13) the appropriate series are substituted for k and k'; these may be obtained directly from (4.3.10) and (2).

Finally, the corresponding expansion for u follows on transforming (13) according to $p \to -p$.

5. ASYMPTOTIC EXPANSIONS OF WEBER PARABOLIC CYLINDER FUNCTIONS $U(\frac{1}{4}a^2, \pm z)$ FOR LARGE a^2.

5.1 Discussion of the solutions.

A detailed account of the definitions and properties of parabolic cylinder functions may be found in Miller (1955). The differential equation satisfied by these functions is written

$$\frac{d^2y}{dz^2} = \frac{1}{4}\left(z^2 + a^2\right) y, \tag{1}$$

and, following Miller, the first solution is taken to be $U(\frac{1}{4}a^2, z)$ defined by its asymptotic behaviour for large z and $|\arg z| < \frac{1}{2}\pi$ as

$$U(\tfrac{1}{4}a^2, z) = e^{-\frac{1}{4}z^2} z^{-\frac{1}{4}a^2 - \frac{1}{2}} \left\{ 1 + O(z^{-2}) \right\}, \quad z \to \infty. \tag{2}$$

The second solution may be taken as $U(\frac{1}{4}a^2, -z)$, and this is a perfectly acceptable solution from a numerical standpoint. Its drawback is that it is not an independent second solution if $-\frac{1}{2}a^2$ is an odd integer. Then this pair gives only one solution, a Hermite function. It is nevertheless possible in this case to define an independent second solution $V(\frac{1}{4}a^2, z)$, which is related to $\mathcal{U}(\frac{1}{4}a^2, z)$ and its mirror image $\mathcal{U}(\frac{1}{4}a^2, -z)$ by

$$V\left(\tfrac{1}{4}a^2, z\right) = \frac{\left(\frac{1}{4}a^2-\frac{1}{2}\right)!}{\pi}\left\{\sin\left(\tfrac{1}{4}\pi a^2\right)\mathcal{U}\left(\tfrac{1}{4}a^2, z\right)+\mathcal{U}\left(\tfrac{1}{4}a^2, -z\right)\right\}. \quad (3)$$

For real z, $\mathcal{U}(\frac{1}{4}a^2, z)$ and $V(\frac{1}{4}a^2, z)$ form a pair of independent solutions for all a^2, but it may prove convenient when $a^2 < 0$ to define as second solution a multiple of $V(\frac{1}{4}a^2, z)$, denoted in Miller (1955, p.17) by $\overline{\mathcal{U}}(\frac{1}{4}a^2, z)$ such that

$$\overline{\mathcal{U}}\left(\tfrac{1}{4}a^2, z\right) = \left(-\tfrac{1}{4}a^2-\tfrac{1}{2}\right)!\ V\left(\tfrac{1}{4}a^2, z\right). \quad (4)$$

The transition points of (1) are at $z = \pm ia$, and the Green-type expansions derived in section 5.3 are immediately valid only for $|\arg z/a| < \frac{1}{2}\pi$. They may, however, easily be extended to cover other domains (but excluding the imaginary axis) by the use of continuation formulae. The more directly relevant of these are (cf. Miller 1955, p.64)

$$U\left(-\tfrac{1}{4}a^2, \pm iz\right) = \frac{\left(\tfrac{1}{4}a^2 - \tfrac{1}{2}\right)!}{\sqrt{2\pi}} \left\{ e^{-i\pi\left(\tfrac{1}{8}a^2 - \tfrac{1}{4}\right)} U\left(\tfrac{1}{4}a^2, \pm z\right) + e^{i\pi\left(\tfrac{1}{8}a^2 - \tfrac{1}{4}\right)} U\left(\tfrac{1}{4}a^2, \mp z\right) \right\}, \tag{5}$$

and

$$U\left(\tfrac{1}{4}a^2, \pm z\right) = \frac{\left(-\tfrac{1}{4}a^2 - \tfrac{1}{2}\right)!}{\sqrt{2\pi}} \left\{ e^{-i\pi\left(\tfrac{1}{8}a^2 + \tfrac{1}{4}\right)} U\left(-\tfrac{1}{4}a^2, \pm iz\right) + e^{i\pi\left(\tfrac{1}{8}a^2 + \tfrac{1}{4}\right)} U\left(-\tfrac{1}{4}a^2, \mp z\right) \right\}. \tag{6}$$

For purposes of identification of our solutions, it is convenient to set down the first few terms in the expansion of $U\left(\tfrac{1}{4}a^2, z\right)$, valid for $a^2 \gg z^2$ and a and z real. From Miller (loc. cit., p.66) these are

$$U\left(\tfrac{1}{4}a^2, z\right) = \frac{\sqrt{\pi}\, e^{-\frac{1}{2}za}}{2^{\frac{1}{8}a^2 + \frac{1}{4}} \left(\tfrac{1}{8}a^2 - \tfrac{1}{4}\right)!} \left\{ 1 - \frac{z^3}{12a} + O\left(a^{-2}\right) \right\}. \tag{7}$$

At $z = 0$, therefore,

$$\mathcal{U}\left(\tfrac{1}{4}a^2, 0\right) = \frac{\pi^{\frac{1}{2}}}{2^{\frac{1}{8}a^2+\frac{1}{4}}\left(\frac{1}{8}a^2-\frac{1}{4}\right)!}. \tag{8}$$

5.2 <u>The integro-differential equation.</u>

In the notation of preceding sections and with reference to (5.1.1), we take

$$\chi = \tfrac{1}{4}\left(z^2 + a^2\right). \tag{1}$$

For large a^2, it follows from (3.2.5) that a pair of solutions, y_1 and y_2 say, of (1) may be written as

$$y_1 = \left(z^2+a^2\right)^{-\frac{1}{4}} \exp\left[\tfrac{1}{4}z\left(z^2+a^2\right)^{\frac{1}{2}} + \tfrac{1}{4}a^2 \sinh^{-1}\left(\frac{z}{\sqrt{a^2}}\right)\right]u, \tag{2}$$

$$y_2 = \left(z^2+a^2\right)^{-\frac{1}{4}} \exp\left[-\tfrac{1}{4}z\left(z^2+a^2\right)^{\frac{1}{2}} - \tfrac{1}{4}a^2 \sinh^{-1}\left(\frac{z}{\sqrt{a^2}}\right)\right]v, \tag{3}$$

where, for instance, $v = 1 + O\left(a^{-2}\right)$. The equation for v is by substitution of (3) into (1), or directly from (3.2.9) and (1),

$$v = \left(z^2 + a^2\right)^{-\frac{1}{2}} \frac{dv}{dz} + \frac{1}{4} \int\int^{z} \left\{ \frac{5}{4} z^2 \left(z^2+a^2\right)^{-\frac{5}{2}} - \frac{1}{2}\left(z^2+a^2\right)^{-\frac{3}{2}} \right\} v \, dz. \qquad (4)$$

A variable, α say, reducing (4) to polynomial form is found with the aid of (3.2.15) . The most convenient choice is with $A = 0$, and $B = \frac{1}{4} a^2$, i.e.

$$\alpha = \frac{z}{\left(z^2 + a^2\right)^{\frac{1}{2}}} \qquad (5)$$

In terms of α , (4) reduces to

$$a^2 v = \left(\alpha^2 - 1\right)^2 v' + \frac{1}{4} \int^{\alpha} \left(5\alpha^2 - 2\right) v \, d\alpha , \qquad (6)$$

which is already in a form suitable for solving by iteration.

5.3 Green-type expansions.

Writing $v = 1 + \sum_{i=1}^{\infty} v_i$, an asymptotic expansion in negative powers of a^2 is obtained from the recurrence relation

$$a^2 v_{i+1} = (\alpha^2 - 1)^2 v_i' + \frac{1}{4} \int^{\alpha} (5\alpha^2 - 2) v_i \, d\alpha . \quad (1)$$

To facilitate identification of the solutions, the lower limit of integration in (1) is chosen such that $v_{i \neq 0} = 0$ at $z = 0$ (i.e. at $\alpha = 0$)+. The first few terms are then calculated to be

$$v_0 = 1 ,$$

$$v_1 = \frac{1}{12a^2} \left(5\alpha^3 - 6\alpha \right) ,$$

$$v_2 = \frac{1}{288 a^4} \left(385\alpha^6 - 924\alpha^4 + 684\alpha^2 \right) ,$$

$$v_3 = \frac{1}{51840 a^6} \left(425425\alpha^9 - 1531530\alpha^7 + 2040012\alpha^5 - 1178280\alpha^3 + 246240\alpha \right),$$

+ In section 6.11 we consider parabolic cylinder functions as special cases of Whittaker functions $W_{k,m}$ with k large, thus obtaining an expansion for v corresponding to a different prescription for the range of integration.

$$V_4 = \frac{1}{2488320a^8}\Big(185910725\,\alpha^{12} - 892371480\,\alpha^{10} + 1722469320\,\alpha^8 - 1683415008\,\alpha^6 + 851530320\,\alpha^4 - 19626240\,\alpha^2\Big),$$

$$V_5 = \frac{1}{208588800\,a^{10}}\Big(188699385875\,\alpha^{15} - 1132196315250\,\alpha^{13} + 2863930469400\,\alpha^{11} - 3940710005232\,\alpha^9 + 3163495898624\,\alpha^7 - 1464916713120\,\alpha^5 + 354807371520\,\alpha^3 - 32972728320\,\alpha\Big). \quad (2)$$

To complete identification of the solutions, it is first noted that at $z=0$,

$$y_2 = a^{-\frac{1}{2}}. \quad (3)$$

Thus by (5.1.8), for large a^2,

$$\mathcal{U}\left(\tfrac{1}{4}a^2, z\right) = \sqrt{\pi a}\,\frac{\left(z^2+a^2\right)^{-\frac{1}{4}}}{2^{\frac{1}{8}a^2+\frac{1}{4}}\left(\frac{1}{8}a^2-\frac{1}{4}\right)!}\,.$$

$$\times \exp\left\{-\tfrac{1}{4}z\left(z^2+a^2\right)^{\frac{1}{2}} - \tfrac{1}{4}a^2\sinh^{-1}\left(\frac{z}{\sqrt{a^2}}\right)\right\}\sum_{i=0}^{\infty} V_i\,. \qquad (4)$$

The second solution y_1, is related to $\mathcal{U}(\frac{1}{4}a^2,-z)$ by the same constants as those relating y_2 and $\mathcal{U}(\frac{1}{4}a^2,z)$. Therefore

$$\mathcal{U}\left(\tfrac{1}{4}a^2, -z\right) = \sqrt{\pi a}\,\frac{\left(z^2+a^2\right)^{-\frac{1}{4}}}{2^{\frac{1}{8}a^2+\frac{1}{4}}\left(\frac{1}{8}a^2-\frac{1}{4}\right)!}\,.$$

$$\times \exp\left\{\tfrac{1}{4}z\left(z^2+a^2\right)^{\frac{1}{2}} + \tfrac{1}{4}a^2\sinh^{-1}\frac{z}{\sqrt{a^2}}\right\}\sum_{i=0}^{\infty}(-1)^i V_i\,. \qquad (5)$$

Similar expansions, but for $a^2 < 0$, follow from (4) and (5) with $\cosh^{-1}$ instead of $\sinh^{-1}$.

5.4 <u>The Mellin transform of V</u>.

The expansions of § 5.3 break down when $z \sim \pm ia$. To obtain expansions for $\mathcal{U}(\frac{1}{4}a^2, \pm z)$ which are valid in this region, and also more generally valid expansions, uniform with respect to z , we apply the Mellin transform technique to (5.2.6).

Let the Mellin transform $M(\mu)$ of v be defined by

$$v = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M(\mu)\, x^{-\mu}\, d\mu\,, \tag{1}$$

with $x = -3\alpha^3/a^2$. The individual terms in (5.2.6) then become, provided no poles are crossed in the translational transformations of μ ,

$$\alpha^4 v' = -\frac{3}{2\pi i}\int_{c-i\infty}^{c+i\infty} M(\mu)\,\mu\left(-\frac{3}{a^2}\right)^{-\mu}\alpha^{-3\mu+3}\, d\mu$$

$$= -\frac{3}{2\pi i}\left(-\frac{3}{a^2}\right)^{-1}\int_{c-i\infty}^{c+i\infty}(\mu+1)\, M(\mu+1)\left(-\frac{3\alpha^3}{a^2}\right)^{-\mu} d\mu\,, \tag{2}$$

$$-2\alpha^2 v' = \frac{6}{2\pi i}\int_{c-i\infty}^{c+i\infty} M(\mu)\,\mu\left(-\frac{3}{a^2}\right)^{-\mu}\alpha^{-3\mu+1}\,d\mu$$

$$= \frac{6}{2\pi i}\left(-\frac{3}{a^2}\right)^{-\frac{1}{3}}\int_{c-i\infty}^{c+i\infty}\left(\mu+\tfrac{1}{3}\right) M\left(\mu+\tfrac{1}{3}\right)\left(-\frac{3\alpha^3}{a^2}\right)^{-\mu} d\mu, \tag{3}$$

$$v' = -\frac{3}{2\pi i}\int_{c-i\infty}^{c+i\infty} M(\mu)\,\mu\left(-\frac{3}{a^2}\right)^{-\mu}\alpha^{-3\mu-1}\,d\mu$$

$$= -\frac{3}{2\pi i}\left(-\frac{3}{a^2}\right)^{\frac{1}{3}}\int_{c-i\infty}^{c+i\infty}\left(\mu-\tfrac{1}{3}\right) M\left(\mu-\tfrac{1}{3}\right)\left(-\frac{3\alpha^3}{a^2}\right)^{-\mu} d\mu, \tag{4}$$

$$\int^{\alpha} \alpha^2 v\, d\alpha = -\frac{1}{6\pi i}\int_{c-i\infty}^{c+i\infty} \frac{M(\mu)}{(\mu-1)}\left(-\frac{3}{a^2}\right)^{-\mu}\alpha^{-3\mu+3}\,d\mu$$

$$= -\frac{1}{6\pi i}\left(-\frac{3}{a^2}\right)^{-1}\int_{c-i\infty}^{c+i\infty} \frac{1}{\mu} M(\mu+1)\left(-\frac{3\alpha^3}{a^2}\right)^{-\mu} d\mu, \tag{5}$$

$$\int^{\alpha} v\,d\alpha = -\frac{1}{6\pi i}\int_{c-i\infty}^{c+i\infty} \frac{M(\mu)}{(\mu-\frac{1}{3})}\left(-\frac{3}{a^2}\right)^{-\mu} \alpha^{-3\mu+1}\, d\mu$$

$$= -\frac{1}{6\pi i}\left(-\frac{3}{a^2}\right)^{-1/3}\int_{c-i\infty}^{c+i\infty} \frac{1}{\mu} M(\mu+\tfrac{1}{3})\left(-\frac{3\alpha^3}{a^2}\right)^{-\mu} d\mu. \qquad (6)$$

Substitution of these quantities into (5.2.6), together with the condition that the sum of the resultant integrals must be zero for all values of $3\alpha^3/a^2$, yields the difference equation

$$(\mu+\tfrac{1}{6})(\mu+\tfrac{5}{6})\,M(\mu+1) - \mu M(\mu) =$$
$$2\left(-\frac{3}{a^2}\right)^{2/3}(\mu+\tfrac{1}{6})^2\, M(\mu+\tfrac{1}{3}) - \left(-\frac{3}{a^2}\right)^{4/3}\mu(\mu-\tfrac{1}{3})\,M(\mu-\tfrac{1}{3}) \qquad (7)$$

which is already in a form suitable for solving by successive approximations. The zero-order approximation to (7), $M_0(\mu)$ say, is obtained by setting the right-hand side equal to zero, i.e. by considering the limit $a^2 \to \infty$. A solution of this difference equation is

$$M_0(\mu) = (\mu-1)!\,\left(-\mu-\tfrac{5}{6}\right)!\,\left(-\mu-\tfrac{1}{6}\right)!\,. \tag{8}$$

Before proceeding with the iteration, it is convenient to remove the zero-order approximation by defining a function $m(\mu)$ such that

$$M(\mu) = M_0(\mu)\, m(\mu)\,. \tag{9}$$

This change of variable transforms (7) into the difference equation

$$m(\mu+1) - m(\mu) = -\left(-\frac{3}{a^2}\right)^{\frac{2}{3}}\left(\mu+\frac{1}{6}\right)\frac{\left(\mu-\frac{2}{3}\right)!\left(-\mu-\frac{1}{2}\right)!}{\left(-\mu-\frac{5}{6}\right)!\,\mu!}\, m\left(\mu+\frac{1}{3}\right)$$

$$-\left(-\frac{3}{a^2}\right)^{\frac{4}{3}}\left(\mu-\frac{1}{6}\right)\frac{\left(\mu-\frac{1}{3}\right)!\left(-\mu-\frac{1}{2}\right)!}{\left(-\mu-\frac{1}{6}\right)!\,(\mu-1)!}\, m\left(\mu-\frac{1}{3}\right), \tag{10}$$

The solution of this equation can be expressed in iterative form as $m(\mu) = \sum_{i=0}^{\infty} m_i(\mu)$, where $m_i(\mu)$ satisfies the sum-equation

$$m_{i+1}(\mu) =$$

$$-\left(-\frac{3}{a^2}\right)^{\frac{2}{3}} \mathcal{S}(\mu+\tfrac{1}{6}) \frac{(\mu-\frac{2}{3})!\,(-\mu-\frac{1}{2})!}{(-\mu-\frac{5}{6})!\quad \mu!}\, m_i(\mu+\tfrac{1}{3})$$

$$-\left(-\frac{3}{a^2}\right)^{\frac{4}{3}} \mathcal{S}(\mu-\tfrac{1}{6}) \frac{(\mu-\frac{1}{3})!\,(-\mu-\frac{1}{2})!}{(-\mu-\frac{1}{6})!\quad (\mu-1)!}\, m_{i-1}(\mu-\tfrac{1}{3}) \cdot \quad (11)$$

Here, contributions have been ordered according to their power in $1/a^2$. The summations which arise are of the type encountered in § 4.3. We take $m_0(\mu) = \left[(-5/6)!\,(-1/6)!\right]^{-1} = 1/2\pi$ to achieve agreement with the Green-type expansion for V given in § 5.3. Performing the summations and combining the results with (8) and (9), we obtain

$$2\pi M(\mu) = (\mu-1)!\,(-\mu-\tfrac{5}{6})!\,(-\mu-\tfrac{1}{6})!$$

$$+\frac{6}{5}\left(-\frac{3}{a^2}\right)^{\frac{2}{3}}(\mu-\tfrac{2}{3})!\,(-\mu-\tfrac{1}{6})!\,(-\mu+\tfrac{1}{2})!$$

$$+\frac{18}{25}\left(-\frac{3}{a^2}\right)^{\frac{4}{3}}\left(\mu^2-\frac{65}{42}\mu+\frac{103}{252}\right)(\mu-\tfrac{1}{3})!\,(-\mu+\tfrac{1}{2})!\,(-\mu-\tfrac{5}{6})!$$

$$+\frac{36}{125}\left(-\frac{3}{a^2}\right)^{2}\left(\mu^5-\frac{65}{14}\mu^4+\frac{2935}{378}\mu^3-\frac{2915}{504}\mu^2+\frac{16823}{9072}\mu+D\right)$$

$$\times(\mu-1)!\,(-\mu-\tfrac{5}{6})!\,(-\mu-\tfrac{1}{6})! + \cdots, \qquad (12)$$

where D is a constant which is to be chosen such that in the domain $|\arg z/a| < \frac{1}{2}\pi$ the Green-type expansion derivable from (1) and (12) agrees with that derived in § 5.3. According to this prescription,

$$D = -\frac{125}{648}.$$

5.5 <u>Transitional expansions</u>.

Combination of (5.4.1) and (5.4.12) leads to an expansion valid in the transitional regions $z \sim \pm ia$ if the contour of integration, chosen such that it separates the poles of $(\mu-1)!$, $(\mu-\frac{2}{3})!$, $(\mu-\frac{1}{3})!$, etc. from those of $(-\mu-\frac{5}{6})!\,(-\mu-\frac{1}{6})!$, $(-\mu-\frac{1}{6})!\,(-\mu+\frac{1}{2})!$,

$\left(-\mu-\frac{5}{6}\right)!\left(-\mu+\frac{1}{2}\right)!$ etc., is closed in a clockwise direction.+ Evaluation of the residues at each pole enclosed yields

$$V = \left(-\tfrac{2}{3}\right)!\left(-\frac{4a^2}{3}\right)^{\frac{1}{6}}(\pi\alpha)^{-\frac{1}{2}}\left[1+\frac{9}{70}\frac{\left(-\frac{1}{6}\right)!}{\sqrt{\pi}}3^{\frac{1}{3}}\left(-\frac{1}{2a^4}\right)^{\frac{2}{3}} - \frac{7}{225a^4}+\cdots\right.$$

$$-\frac{\left(-\frac{1}{6}\right)!}{2\sqrt{\pi}\alpha^2}3^{\frac{1}{3}}\left(-\tfrac{1}{2}a^2\right)^{\frac{2}{3}}\left\{1-\frac{4}{5}\frac{\left(\frac{1}{6}\right)!}{\sqrt{\pi}}3^{\frac{2}{3}}\left(-\frac{1}{2a^4}\right)^{\frac{1}{3}}-\frac{32}{1575a^4}\cdots\right\}$$

$$-\frac{a^2}{6\alpha^3}\left\{1+\frac{9}{70}\frac{\left(-\frac{1}{6}\right)!}{\sqrt{\pi}}\left(-\frac{1}{2a^4}\right)^{\frac{2}{3}}-\frac{7}{225a^4}\cdots\right\}$$

$$-\frac{\left(-\frac{1}{6}\right)!}{2\sqrt{\pi}\alpha^4}3^{\frac{1}{3}}\left(-\tfrac{1}{2}a^2\right)^{\frac{2}{3}}\left\{1-\frac{17}{40}\frac{\left(\frac{1}{6}\right)!}{\sqrt{\pi}}3^{\frac{2}{3}}\left(-\frac{1}{2a^4}\right)^{\frac{1}{3}}+\cdots\right\}$$

$$\left.+\cdots\right], \quad (1)$$

with $\alpha = z/(z^2+a^2)^{\frac{1}{2}}$.

The series for u in (5.2.2) is deduced directly from (1) on replacing α with $-\alpha$.

\+ The Green-type expansion (5.3.2) is regained on closing the contour of integration thus specified in an anti-clockwise direction, and evaluating the residues at the enclosed poles, i.e. at the poles of $(\mu-1)!$, $(\mu-\frac{2}{3})!$, etc.

5.51 Expansions at the transition points.

The asymptotic expansion of $\mathcal{U}(\frac{1}{4}a^2, z)$ precisely at the transition point $z = ia$ is

$$\mathcal{U}\left(\tfrac{1}{4}a^2, ia\right) = \frac{(-\frac{2}{3})!\; e^{-\frac{1}{4}i\pi(1+\frac{1}{2}a^2)}}{(2\pi)^{\frac{1}{4}}\left\{(\frac{1}{4}a^2-\frac{1}{2})!\right\}^{\frac{1}{2}}} \left(\frac{3ia}{32}\right)^{-\frac{1}{6}}$$

$$\times\left\{1 + \frac{9}{70}\,\frac{(-\frac{1}{6})!}{\sqrt{\pi}}\,3^{\frac{1}{3}}\left(-\frac{1}{2a^4}\right)^{\frac{2}{3}} - \frac{7}{225a^4} + \cdots\right\}. \quad (1)$$

A similar expansion for $\mathcal{U}(\frac{1}{4}a^2, -ia)$ follows by analytic continuation.

5.6 Uniform expansions involving modified Bessel functions of order one-third.+

Expansions are sought which are valid uniformly with respect to z. The subsidiary function $k(x)$ will, as in § 4.5, be taken to be proportional to the inverse Mellin transform of $M_0(\mu)$. Thus

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+ Uniform expansions for parabolic cylinder functions involving Airy functions have been discussed by Olver (1959, Part 3).

$$k(x) = \frac{1}{4\pi^2 i} \int_{c-i\infty}^{c+i\infty} (\mu-1)!\left(-\mu-\tfrac{5}{6}\right)!\left(-\mu-\tfrac{1}{6}\right)!\, x^{-\mu}\, d\mu . \quad (1)$$

This is the same subsidiary function as that selected in § 4.5, except that here $x = -3\alpha^3/a^2$. The properties of $k(x)$ and its derivates have already been examined and discussed in detail in § 4.5, and by the method of that section (5.4.1) and (5.4.12) yield the following uniform expansion for V :

$$V = k(x)\left\{1 + \frac{\alpha}{a^2}\sum_{n=1}^{\infty} a^{2n}\, a_n(x)\, \alpha^{-5n}\right\}$$

$$+ \frac{18\alpha}{5a^2}\, k'(x) \sum_{n=0}^{\infty} a^{2n}\, b_n(x)\, \alpha^{-5n},$$

(2)

where the first few coefficients are

$$a_1(x) = \frac{1}{140}\left(x^2 + 3x - 14\right),$$

$$a_2(x) = \frac{1}{56700}\left(71x^3 - 64x^2 + 2538x + 756\right),$$

$$b_0(x) = 1,$$

$$b_1(x) = \frac{1}{420}\left(5x^3 - 150x - 84\right),$$

$$b_2(x) = \frac{1}{510300}\left(85x^4 - 1215x^3 + 6110x^2 + 72900x + 13608\right). \quad (3)$$

The corresponding expansion for u follows on replacing $+\alpha$ by $-\alpha$.

(To recover the Green-type and transitional expansions from (2), it is only necessary to substitute the homologous expansions for $k(x)$ and $k'(x)$. These may be conveniently obtained from their Mellin-Barnes integral representations (4.3.10) and (4.5.2), respectively.)

The full expansions for $\mathcal{U}\left(\frac{1}{4}a^2, \pm z\right)$ include, of course, the Liouville-Green factors quoted in (5.3.4) and (5.3.5).

6. ASYMPTOTIC EXPANSIONS OF WHITTAKER FUNCTIONS $W_{k,m}$ FOR LARGE k, AND FOR LARGE $|k^2-m^2|$.

6.1 Green-type expansions for Whittaker functions; k large, m small.

Comprehensive accounts of Whittaker functions may be found in Whittaker & Watson (1927, ch. 16), Buchholz (1953) and Slater (1960).

The functions $W_{k,m}$ satisfy the differential equation

$$\frac{d^2W}{dz^2} \doteq \left(\frac{1}{4} - \frac{k}{z} + \frac{m^2-\frac{1}{4}}{z^2}\right) W. \tag{1}$$

A convenient first solution is defined by

$$W_{k,m}(z) = e^{-\frac{1}{2}z} z^k \left\{1 + O\left(z^{-1}\right)\right\}; \quad |z| \gg |k^2|, \quad |\arg z| < \pi, \tag{2}$$

Since (1) is unaltered by a simultaneous reversal in sign of k and z, another solution is $W_{-k,m}(-z)$. This proves to be a particularly satisfactory pair of solutions in that the Wronskian of $W_{k,m}(z)$ and

$W_{-k,m}(-z)$ never vanishes for any values of k and m (cf. Slater 1960, p.26; for relation to generalized Laguerre functions, see also p.95 of this reference).

The results that follow in this section may be made to cover all ranges of $z/4k$ excepting the immediate neighbourhood of the point $z = 4k$. The reason for this exception becomes apparent shortly; further restrictions on z and k will be quoted after particular expansions have been derived.

Equation (1) is already of the form $d^2W/dz^2 = \chi W$, but direct application of any W.K.B. method leads to complications in the higher terms, in that these would then contain logarithmic as well as algebraic quantities. This difficulty originates in the singularity occurring with this choice of χ at $z = 0$, and can be avoided (cf. Langer 1937) by a transformation, involving both dependent and independent variables, studied particularly by Kamke (1944, p.476). With

$$z = e^{x} \quad , \quad W = e^{\frac{1}{2}x} \mathcal{R} , \qquad (3)$$

equation (1) becomes

$$\frac{d^2 \mathcal{R}}{d x^2} = \left(\tfrac{1}{4} e^{2x} - k e^{x} + m^2 \right) \mathcal{R} , \tag{4}$$

and there is no longer a singularity at the (new) origin. Since in this section m^2 is taken to be small, χ can be split up into a large and a small part; in the notation of § 3.2

$$\mathcal{E} = m^2 , \; K = \tfrac{1}{4} e^{2x} - k e^{x} , \tag{5}$$

and solutions of (4) may then be written as in (3.2.13). With the original variable and the second relation in (3), a pair of independent solutions of (1) is

$$y_1 = \left(1 - \frac{4k}{z}\right)^{-\frac{1}{4}} \exp\left\{ \tfrac{1}{2} z \left(1 - \frac{4k}{z}\right)^{\frac{1}{2}} - 2k \cosh^{-1}\left(\frac{z}{4k}\right)^{\frac{1}{2}} \right\} u, \tag{6}$$

$$y_2 = \left(1 - \frac{4k}{z}\right)^{-\frac{1}{4}} \exp\left\{ -\tfrac{1}{2} z \left(1 - \frac{4k}{z}\right)^{\frac{1}{2}} + 2k \cosh^{-1}\left(\frac{z}{4k}\right)^{\frac{1}{2}} \right\} v, \tag{7}$$

with v satisfying the integro-differential equation

$$V = z^{\frac{1}{2}}\left(z-4k\right)^{-\frac{1}{2}}\frac{dv}{dz} + \frac{1}{4}\int^{z} z^{-\frac{3}{2}}\left\{\frac{z^2+4k^2}{(z-4k)^{5/2}} - \frac{4m^2}{(z-4k)^{1/2}}\right\} v\,dz. \tag{8}$$

A suitable new variable reducing (8) to polynomial-like form is suggested by (3.2.15), K now replacing γ . With, for instance, $A = 0$ and $B = \frac{1}{2}$ this gives

$$\frac{z^{\frac{1}{2}}}{(z-4k)^{\frac{1}{2}}} - \frac{5k}{z^{\frac{1}{2}}(z-4k)} , \tag{9}$$

but it proves sufficient and more convenient to take as new variable, t say, only the first term

$$t = \frac{z^{\frac{1}{2}}}{(z-4k)^{\frac{1}{2}}} . \tag{10}$$

In terms of t , (8) contracts to the polynomial-type relation

$$-8kv = \left(t^2-1\right)^2 \frac{dv}{dt} + \frac{1}{4}\int^t \left(5t^2 - 2 - \frac{\eta}{t^2}\right) v\,dt\,, \tag{11}$$

where $\eta = 16m^2 - 1$. Writing $v = \sum_{i=0}^{\infty} v_i$,

successive v_i's are given by the recurrence relation

$$-8kv_{i+1} = \left(t^2-1\right)^2 \frac{dv_i}{dt} + \frac{1}{4}\int^t \left(5t^2 - 2 - \frac{\eta}{t^2}\right) v_i\,dt\,. \tag{12}$$

The simultaneous appearance of both positive and negative powers of t in (12) complicates the problem of choosing suitable lower limits of integration to give precisely identifiable solutions, since zero and infinity are then precluded as possible points of identification. Examination shows that the simplest possible expansion will be that for which when i is odd, v_i contains only odd powers of t and no constants, i.e. is completely odd in t ; and for which when i is even, v_i is completely even in t .

Starting with $V_0 = 1$, direct calculation from (12) shows that

$$V_1 = -\frac{1}{96k}\left(5t^3 - 6t + \frac{3m}{t} + \text{constant}\right), \qquad (13)$$

and in accordance with the prescription just given, the constant will be taken equal to zero to keep V_1 odd in t . The choice of subsequent integration constants leading to precisely identifiable expansions can be achieved by considering the particular case $t = 1$. According to the choice made so far,

$$V(1) = 1 - \frac{3m-1}{96k} + O\left(k^{-2}\right). \qquad (14)$$

By (10), $t = 1$ corresponds to the limit $|z| >> |k|$: an expansion for $W_{k,m}(z)$ in this limit is derived in appendix A. This expansion in inverse powers of k , (A6), also starts essentially as $1 - (3m-1)/96k + O(k^{-2})$, and it is thuss convenient to identify the successive integration constants in (12) such that $V(1)$ shall agree precisely with this expansion (A6). Now for

$|z| >> |k^2|$, expansion (A6) and that given in Whittaker and Watson (1927, § 16.3) --- valid, if m is small, for $|z| > |k^2|$ --- must tend to the same limit, that is to (2). Furthermore, from (7),

$$y_2 \sim e^{-\frac{1}{2}z} z^k e^{-k} k^k v(1), \quad |z| >> |k|, \tag{15}$$

so that, identifying y_2 with $W_{k,m}(z)$ and comparing (15) with (A6), it follows that

$$v(1) = \frac{(2\pi)^{\frac{1}{2}} e^{-k} k^k}{\left\{(k+m-\frac{1}{2})!(k-m-\frac{1}{2})!\right\}^{\frac{1}{2}}} . \tag{16}$$

$v(1)$ is now expanded as a series in negative powers of k using Stirling's series in the form

$$(p+h-1)! = (2\pi)^{\frac{1}{2}} p^{p+h-\frac{1}{2}} e^{-p} \exp\left\{\sum_{r=1}^{\infty} (-1)^{r-1} \frac{B_{r+1}(h)}{r(r+1)\, p^r}\right\}; \tag{17}$$

here $B_r(h)$ is the rth Bernoulli polynomial as defined in Nörlund (1923, p.19) and Milne-Thomson (1933, pp.136-137). Substitution for p and h in (17) yields

$$v(1) = 1 - \frac{3\eta-1}{96k} + \frac{9\eta^2-6\eta+1}{18\,432\,k^2} - \frac{135\eta^3+8505\eta^2-51795\eta+4027}{26\,542\,080\,k^3} + \frac{405\eta^4+103140\eta^3-656370\eta^2+255684\eta-16123}{10\,192\,158\,720\,k^4} - \cdots . \tag{18}$$

With the lower limits of integration in (12) chosen such that for $t = 1$ the terms in the expansion for v correspond to those in (18), we find for the first six contributions

$$v_0 = 1 ,$$

$$V_i = \frac{\alpha_i}{(-k)^i} \sum_{r=0}^{2i} a_{ri}\, t^{3i-2r}, \qquad (19)^+$$

where

$$\alpha_1 = \frac{1}{96}\ ;\ a_{01} = 5,\ a_{11} = -6,\ a_{21} = 3\eta\ ;$$

$$\alpha_2 = \frac{1}{18432}\ ;\ a_{02} = 385,\ a_{12} = -924,\ a_{22} = -6(7\eta - 114),$$

$$a_{32} = 36(3\eta - 4),\ a_{42} = 9\eta(\eta - 8)\ ;$$

$$\alpha_3 = \frac{1}{26\,542\,080}\ ;\ a_{03} = 425\,425,\ a_{13} = -1\,531\,530,$$

$$a_{23} = -20475\eta + 2\,040\,012,$$

$$a_{33} = 53460\eta - 1\,189\,080,$$

+ V_1 and V_2 are in agreement with the corresponding terms derived by Buchholz (1953, p.103) using the method of steepest descents, though his multiplying constant is inexact since contributions due to $B_4(m+\frac{1}{2})$, $B_6(m+\frac{1}{2})$, etc., are not taken into account --- he approximates $V(1)$ by $\exp\{-B_2(m+\frac{1}{2})/2k\} = \exp\{-(4m^2-\frac{1}{2})/8k\}$.

$$a_{43} = 675\,\eta^2 - 45\,900\,\eta + 259\,200\,,$$

$$a_{53} = 12150\,\eta^2 - 64\,800\,\eta\,,$$

$$a_{63} = 135\,\eta^3 - 4320\,\eta^2 + 25\,920\,\eta\,;$$

$$\alpha_4 = \frac{1}{10\,192\,158\,720}\,; \quad a_{04} = 185\,910\,725\,,$$

$$a_{14} = -892\,371\,480\,,$$

$$a_{24} = -5\,705\,700\,\eta + 1722\,469\,320\,,$$

$$a_{34} = 21\,205\,800\,\eta - 1\,685\,078\,208\,,$$

$$a_{44} = 103\,952\,\eta^2 - 29\,642\,976\,\eta + 855\,522\,000,$$

$$a_{54} = -612\,360\,\eta^2 + 20\,956\,320\,\eta - 199\,221\,120\,,$$

$$a_{64} = 22\,140\,\eta^3 + 1\,420\,200\,\eta^2 - 13\,711\,680\,\eta + 12\,752\,640\,,$$

$$a_{74} = 113\,400\,\eta^3 - 2\,268\,000\,\eta^2 + 10\,886\,400\,\eta,$$

$$a_{84} = 405\,\eta^4 - 32\,400\,\eta^3 + 699\,840\,\eta^2 - 3\,732\,480\,\eta;$$

$$\alpha_5 = \frac{1}{34\,245\,653\,299\,200},$$

$$a_{05} = 943\,496\,929\,375, \quad a_{15} = 5\,660\,981\,576\,250,$$

$$a_{25} = -19\,443\,469\,500\,\eta + 1\,521\,285\,685\,920,$$

$$a_{35} = 103\,562\,058\,600\,\eta - 19\,707\,838\,310\,160,$$

$$a_{45} = 268\,021\,260\,\eta^2 - 204\,545\,020\,680\,\eta$$

$$+ 15\,832\,284\,539\,760,$$

$$a_{55} = -1\,377\,646\,704\,\eta^2 + 209\,474\,385\,120\,\eta$$

$$- 7\,345\,146\,886\,560,$$

$$a_{65} = 236\,180\,\eta^3 + 1\,503\,543\,720\,\eta^2$$

$$- 83\,609\,956\,080\,\eta + 1\,453\,454\,150\,400,$$

$$a_{75} = 23\,927\,400\,\eta^3 - 2\,298\,618\,000\,\eta^2$$

$$+ 37\,970\,763\,200\,\eta - 170\,023\,795\,200,$$

$$a_{85} = 1\,431\,675\,\eta^4 + 434\,889\,000\,\eta^3$$

$$- 7\,269\,393\,600\,\eta^2 + 30\,144\,441\,600\,\eta,$$

$$a_{95} = 5\,337\,150\,\eta^4 - 285\,768\,000\,\eta^3$$

$$+ 4\,686\,595\,200\,\eta^2 - 21\,946\,982\,400\,\eta,$$

$$a_{105} = 8505\,\eta^5 - 1\,360\,800\,\eta^4 + 69\,128\,640\,\eta^3$$

$$- 1\,254\,113\,280\,\eta^2 + 6\,270\,566\,400\,\eta. \qquad (20)$$

Identification of y_2 with $W_{k,m}(z)$ is now complete, and we have explicitly

$$W_{k,m}(z) = \frac{\left\{\left(k+m-\frac{1}{2}\right)!\left(k-m-\frac{1}{2}\right)!\right\}^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}}\left(1-\frac{4k}{z}\right)^{-\frac{1}{4}}$$

$$\times \exp\left\{-\frac{1}{2}z\left(1-\frac{4k}{z}\right)^{\frac{1}{2}} + 2k\cosh^{-1}\left(\frac{z}{4k}\right)^{\frac{1}{2}}\right\}\sum_{i=0}^{\infty} V_i. \quad (21)$$

Similarly,

$$W_{-k,m}\left(ze^{\pm i\pi}\right) = \frac{e^{\mp i\pi k}(2\pi)^{\frac{1}{2}}}{\left\{(k+m-\frac{1}{2})!\,(k-m-\frac{1}{2})!\right\}^{\frac{1}{2}}}\left(1-\frac{4k}{z}\right)^{-\frac{1}{4}}$$

$$\times \exp\left\{\tfrac{1}{2}z\left(1-\frac{4k}{z}\right)^{\frac{1}{2}} - 2k\cosh^{-1}\left(\frac{z}{4k}\right)^{\frac{1}{2}}\right\}\sum_{i=0}^{\infty}(-1)^i v_i, \quad (22)$$

where

$-\frac{1}{2}\pi \leq (\arg z, \arg k) \leq 0$ for the upper signs,

$0 \leq (\arg z, \arg k) \leq \frac{1}{2}\pi$ for the lower signs.

Ranges of applicability.

For the sake of brevity we will illustrate the ranges of applicability of the foregoing results only for the important special case $\arg z = \arg k$. Then there are four cases to consider:

(i) In the range $1 < z/4k < \infty$, t is always real and less than unity, and the expansions for $W_{k,m}(z)$ and $W_{-k,m}\left(ze^{\pm i\pi}\right)$ as given by (21) and (22) may be applied as they stand.

(ii) For $0 < z/4k < 1$, t is imaginary and it suffices to change the variable t to t', where $t = t' e^{\pm\frac{1}{2}i\pi}$ depending on the phase of z, giving an expansion for $W_{-k,m}(z e^{\pm i\pi})$ from which, by a continuation formula given in, for example, Slater (1960, § 2.5.2), the corresponding expansion for $W_{k,m}(z)$ follows. (See further under Poiseuille functions in § 6.5.)

(iii) Results applicable in the range $-\infty < \frac{z}{4k} < 0$ are conveniently deduced by reversing the sign of k in (21) giving an expansion for $W_{-k,m}(z)$. From this expansion with $z \to z e^{-2i\pi}$, it is then possible to obtain an expansion for $W_{k,m}(z e^{-i\pi})$ by means of a known continuation formula (Slater 1960, (2.5.19)).

(iv) From the foregoing work it is not possible to derive any results valid for $z \sim 4k$, since in this transitional region $\chi^{-\frac{1}{4}}$ (or $K^{-\frac{1}{4}}$) is no longer bounded and slowly varying, and the step from (11) to (12) results in a rapidly diverging series for v. A more powerful technique successfully covering this case will be applied in § 6.3.

6.11 <u>Parabolic cylinder functions</u>.

Alternative expansions for parabolic cylinder functions to those given in section 5.3 may be deduced from (6.1.21) and (6.1.22). From Miller (1955, p.73) we obtain the relation

$$\mathcal{U}\left(\tfrac{1}{4}a^2, z\right) = 2^{-\frac{1}{8}a^2} z^{-\frac{1}{2}} W_{-\frac{1}{8}a^2, -\frac{1}{4}}\left(\tfrac{1}{2}z^2\right). \qquad (1)$$

The appropriate substitutions in (6.1.21) and (6.1.22) are therefore

$$k = -\tfrac{1}{8}a^2, \; \eta = 0, \; t = \alpha. \qquad (2)$$

By the continuation formula +

$$W_{-k,m}(x) = \frac{(m-k-\frac{1}{2})!\,(-m-k-\frac{1}{2})!}{2\pi i} \cdot \times \left\{ e^{-\pi i k} W_{k,m}\left(xe^{\pi i}\right) - e^{\pi i k} W_{k,m}\left(xe^{-\pi i}\right) \right\}, \qquad (3)$$

quoted in e.g. Slater (1960, (2.5.19)), the alternative expansions for real a are

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+ Instead, we could have proceeded via (A5).

$$U\left(\tfrac{1}{4}a^2, z\right) = \frac{(2\pi)^{\frac{1}{4}}\left(z^2+a^2\right)^{-\frac{1}{4}}}{\left\{\left(\frac{1}{4}a^2-\frac{1}{2}\right)!\right\}^{\frac{1}{2}}} \exp\left\{-\frac{z}{4}\left(z^2+a^2\right)^{\frac{1}{2}} - \frac{a^2}{4}\sinh^{-1}\left(\frac{z}{\sqrt{a^2}}\right)\right\}$$

$$\times \sum_{i=0}^{\infty} V_i \quad , \qquad (4)$$

$$U\left(\tfrac{1}{4}a^2, -z\right) = \frac{(2\pi)^{\frac{1}{4}}\left(z^2+a^2\right)^{-\frac{1}{4}}}{\left\{\left(\frac{1}{4}a^2-\frac{1}{2}\right)!\right\}^{\frac{1}{2}}} \exp\left\{\frac{z}{4}\left(z^2+a^2\right)^{\frac{1}{2}} + \frac{a^2}{4}\sinh^{-1}\left(\frac{z}{\sqrt{a^2}}\right)\right\}$$

$$\times \sum_{i=0}^{\infty} (-1)^i V_i \quad . \qquad (5)$$

Substituting (2) into (6.1.20), we obtain for the first few terms

$$V_0 = 1 \quad ,$$

$$V_1 = \frac{1}{12a^2}\left(5\alpha^3 - 6\alpha\right),$$

$$V_2 = \frac{1}{288\, a^4}\left(385\,\alpha^6 - 924\,\alpha^4 + 684\,\alpha^2 - 144\right),$$

$$V_3 = \frac{1}{51840\, a^6}\left(425\,425\,\alpha^9 - 1\,531\,530\,\alpha^7 + 2\,040\,012\,\alpha^5 - 1\,189\,080\,\alpha^3 + 259\,200\,\alpha\right),$$

$$V_4 = \frac{1}{2\,488\,320\, a^8}\left(185\,910\,725\,\alpha^{12} - 892\,371\,480\,\alpha^{10} + 1\,722\,469\,320\,\alpha^8 - 1\,685\,078\,208\,\alpha^6 + 855\,522\,000\,\alpha^4 - 199\,221\,120\,\alpha^2 + 12\,752\,640\right),$$

$$V_5 = \frac{1}{209\,018\,880\, a^{10}}\left(188\,699\,385\,875\,\alpha^{15} - 1\,132\,196\,315\,250\,\alpha^{13} + 3\,042\,257\,137\,184\,\alpha^{11} - 3\,941\,567\,662\,032\,\alpha^9 + 3\,166\,456\,907\,952\,\alpha^7 - 1\,469\,029\,377\,312\,\alpha^5 + 290\,690\,830\,080\,\alpha^3 - 34\,004\,759\,040\,\alpha\right). \quad (6)$$

Analogous expressions for $\alpha^2 < 0$ follow by analytic continuation, or directly from (6.1.21) and (6.1.22).

Expansions of the same character have also been derived by Darwin (1949) and by Olver (1959).

6.2 Green-type expansions for $W_{k,m}(z)$; $|k^2 - m^2|$ large.

For brevity it will temporarily be assumed that $k^2 > m^2$ and that z, k and m are all real. Actually, the results will be applicable for less severe restrictions; some of these will be discussed after the expansions have been derived; in addition explicit expansions will then be stated for the case $k^2 < m^2$.

Since m^2 can no longer be considered to be small, it is necessary to retain the full expression for χ. With the aid of Kamke's form (6.1.4), which gives

$$\chi = \tfrac{1}{4} e^{2x} - k e^{x} + m^2, \tag{1}$$

a pair of independent solutions of (6.1.1) may be written

$$y_1 = \left(1 - \frac{4k}{z} + \frac{4m^2}{z^2}\right)^{-\frac{1}{4}}$$

$$\times \exp\left\{\frac{z}{2}\left(1 - \frac{4k}{z} + \frac{4m^2}{z^2}\right)^{\frac{1}{2}} - k \cosh^{-1}\left(\frac{z-2k}{2\kappa}\right) + |m| \cosh^{-1}\left(\frac{kz-2m^2}{\kappa z}\right)\right\} u, \tag{2}$$

$$y_2 = \left(1 - \frac{4k}{z} + \frac{4m^2}{z^2}\right)^{-\frac{1}{4}}$$

$$\times \exp\left\{-\frac{z}{2}\left(1 - \frac{4k}{z} + \frac{4m^2}{z^2}\right)^{\frac{1}{2}} + k \cosh^{-1}\left(\frac{z-2k}{2\kappa}\right) - |m| \cosh^{-1}\left(\frac{kz-2m^2}{\kappa z}\right)\right\} v, \tag{3}$$

with $\kappa = (k^2 - m^2)^{\frac{1}{2}}$. The integro-differential equation satisfied by v is

$$v = \frac{z \, dv/dz}{(z^2 - 4kz + 4m^2)^{\frac{1}{2}}} + \frac{1}{4}\int^z \frac{z^3 + 4(k^2 - 4m^2)z + 16km^2}{(z^2 - 4kz + 4m^2)^{5/2}} v \, dz. \tag{4}$$

A suitable new variable reducing (4) to polynomial-like form is suggested by (3.2.15). With, for instance, $A = 0$, $B = \frac{1}{2}\kappa^2$, this would be

$$\frac{2(kz - m^2 + \kappa)}{(z^2 - 4kz + 4m^2)^{1/2}} + \left\{\frac{z - 2k + 2\kappa}{z - 2k - 2\kappa}\right\}^{\frac{1}{2}}, \tag{5}$$

but it proves sufficient and more convenient to take as new variable, y say, only the second term

$$y = \left\{ \frac{z - 2k + 2\kappa}{z - 2k - 2\kappa} \right\}^{\frac{1}{2}}. \tag{6}$$

This choice is especially favourable in that it reduces to t (the variable selected in § 6.1) when $|k| \gg |m|$. In terms of y, (4) contracts to

$$-16\kappa v = \left(Ay^4 - By^2 - b + \frac{a}{y^2}\right)\frac{dv}{dy} + \frac{1}{4}\int^y \left(5Ay^2 - B - \frac{b}{y^2} + \frac{5a}{y^4}\right) v \, dy, \tag{7}$$

where

$$A = 1+\sigma,\ a = 1-\sigma,\ B = 1+3\sigma,\ b = 1-3\sigma,\ \sigma = k/\kappa. \tag{8}$$

The recurrence relation becomes

$$-16\kappa v_{i+1} = \left(Ay^4 - By^2 - b + \frac{a}{y^2}\right)\frac{dv_i}{dy} + \frac{1}{4}\int^y \left(5Ay^2 - B - \frac{b}{y^2} + \frac{5a}{y^4}\right) v_i \, dy. \tag{9}$$

On iteration this yields a series for V in inverse powers of κ. The integration constants in (7) and (9) will be chosen such that for $|k| >> |m|$ the series goes over into (6.1.19). Thus, from (6.1.16), in the limit $|z| >> |\kappa|$, that is with $y = 1$,

$$V(1) = \frac{f(k,m)}{\left\{(k+m-\frac{1}{2})!\,(k-m-\frac{1}{2})!\right\}^{\frac{1}{2}}}, \tag{10}$$

$f(k,m)$ being a function of the parameters k and m only. $V(1)$ is now expanded as a series in inverse powers of κ, again with the use of Stirling's formula in the form (6.1.17). This leads to

$$V(1) = (2\pi)^{-\frac{1}{2}} f(k,m)\, e^{k} (k+m)^{-\frac{1}{2}(k+m)} (k-m)^{-\frac{1}{2}(k-m)}$$

$$\times \exp\left[-\frac{1}{2}\sum_{r=1,3,5,\ldots}^{\infty} \frac{B_{r+1}(\frac{1}{2})}{r(r+1)\,\kappa^{2r}}\left\{(k+m)^{r} + (k-m)^{r}\right\}\right]$$

$$= (2\pi)^{-\frac{1}{2}} f(k,m)\, e^{k} (k+m)^{-\frac{1}{2}(k+m)} (k-m)^{-\frac{1}{2}(k-m)}$$

$$\times\left\{1 + \frac{k}{24\,\kappa^{2}} + \frac{k}{1152\,\kappa^{4}} - \frac{1003\,k^{3} + 3024\,m^{2}k}{414\,720\,\kappa^{6}} + \frac{k^{4}}{7962624\,\kappa^{8}} + \cdots\right\}. \tag{11}$$

The condition $v_0 = 1$, taken together with the fact that direct calculation from (9) gives $v_1 = k/24\,\kappa^2$, determines the function $f(k,m)$. The result is

$$v(1) = (2\pi)^{\frac{1}{2}} \frac{e^{-k} (k+m)^{\frac{1}{2}(k+m)} (k-m)^{\frac{1}{2}(k-m)}}{\left\{ (k+m-\frac{1}{2})!\,(k-m-\frac{1}{2})! \right\}^{\frac{1}{2}}} , \tag{12}$$

With the above prescription, the polynomials v_i may be written

$$v_i = \frac{\beta_i}{(-\kappa)^i} \sum_{r=0}^{3i} b_{ri}\, y^{3i-2r} . \tag{13}$$

The first few coefficients are calculated to be

$$\beta_1 = \frac{1}{192} \;;\quad b_{01} = 5A,\; b_{11} = -3B,\; b_{21} = 3b,\; b_{31} = -5a;$$

$$\beta_2 = \frac{1}{73728} \;;\quad b_{02} = 385\,A^2,\; b_{12} = -462\,AB,$$

$$b_{22} = 3(27\,B^2 - 134\,Ab),$$

$$b_{32} = 4(199 - 451\,\sigma^2),\ b_{42} = 3(27 b^2 - 134\, aB),$$

$$b_{52} = -462\, ab,\quad b_{62} = 385\, a^2;$$

$$\beta_3 = \frac{1}{212\,336\,640};$$

$$b_{03} = 425\,425\ A^3,\quad b_{13} = -765\,765\ A^2 B,$$

$$b_{23} = 369\,603\ AB^2 - 582\,075\ A^2 b,$$

$$b_{33} = 490\,320\ ABb - 30\,375\ B^3 + 444\,675\ aA^2$$
$$+ 59700\ A - 135\,300\ A\sigma^2,$$

$$b_{43} = 151\,875\ Ab^2 - 382\,320\ AaB - 32\,805\ bB^2$$
$$- 35820\ B + 81\,180\ B\sigma^2,$$

$$b_{53} = -(151\,875\ aB^2 - 382\,320\ aAb - 32\,805\ Bb^2$$
$$- 35\,820\ b + 81\,180\ b\sigma^2),$$

$$b_{63} = -\left(490\,320\, abB - 30375\, b^3 + 444\,675\, Aa^2 + 59700\, a - 135\,300\, a\sigma^2\right),$$

$$b_{73} = -\left(369\,603\, ab^2 - 582\,075\, a^2 B\right),\ b_{83} = 765\,765\, a^2 b,$$

$$b_{93} = -425\,425\, a^3\ ;$$

$$\beta_4 = \frac{1}{163\,074\,539\,520}\ ;$$

$$b_{04} = 185\,910\,725\ A^4\ ,\ b_{14} = -446\,185\,740\, A^3 B,$$

$$b_{24} = 349\,922\,430\ A^2 B^2 - 328\,485\,300\ A^3 b,$$

$$b_{34} = 475\,200\,000\ A^2 Bb - 94\,121\,676\ AB^3 + 256\,517\,800\, aA^3 + 9\,193\,800\ A^2 - 20\,836\,200\ A^2\sigma^2,$$

$$b_{44} = 151\,012\,350\, A^2 b^2 - 335\,263\,950\, A^2 a B$$

$$-163\,957\,284\, A b B^2 - 11\,032\,560\, A B$$

$$+25\,003\,440\, A B \sigma^2 + 4\,465\,125\, B^4 ,$$

$$b_{54} = 123\,171\,840\, a A B^2 - 230\,743\,260\, a b A^2$$

$$-82\,339\,740\, A B b^2 - 9\,599\,760\, A b$$

$$+21\,756\,240\, A b \sigma^2 + 6\,327\,720\, b B^3 + 1\,934\,280\, B^2$$

$$-4\,383\,720\, B^2 \sigma^2 ,$$

$$b_{64} = 276\,146\,400 - 2\,816\,527\,920 \sigma^2$$

$$+2\,986\,179\,600\, \sigma^4 ,$$

$$b_{74} = 123\,171\,840\, A a b^2 - 230\,743\,260\, A B a^2$$

$$-82\,339\,740\, a b B^2 - 9\,599\,760\, a B$$

$$+21\,756\,240\, a B \sigma^2 + 6\,327\,720\, B b^3$$

$$+ 1934\,280\, b^2 - 4\,383\,720\, b^2 \sigma^2 ,$$

$$b_{84} = 151\,012\,350\, a^2 B^2 - 335\,263\,950\, a^2 A b$$

$$- 163\,957\,284\, a B b^2 - 11\,032\,560\, a b$$

$$+ 25\,003\,440\, a b \sigma^2 + 4\,465\,125\, b^4 ,$$

$$b_{94} = 475\,200\,000\, a^2 b B - 94\,121\,676\, a b^3$$

$$+ 256\,517\,800\, A a^3 + 9\,193\,800\, a^2$$

$$- 20\,836\,200\, a^2 \sigma^2 ,$$

$$b_{104} = 349\,922\,430\, a^2 b^2 - 328\,485\,300\, a^3 B ,$$

$$b_{114} = - 446\,185\,740\, a^3 b ,$$

$$b_{124} = 185\,910\,725\, a^4 . \qquad (14)$$

It remains to complete identification of the solutions. In the limit $|z| >> |\kappa|$, it follows from (3) and (12) that

$$y_2 \sim (2\pi)^{\frac{1}{2}} \frac{e^{-\frac{1}{2}z} z^{k}}{\left\{(k+m-\frac{1}{2})!\,(k-m-\frac{1}{2})!\right\}^{\frac{1}{2}}}; \tag{15}$$

and from Whittaker & Watson (1927, § 16.3)

$$W_{k,m}(z) = e^{-\frac{1}{2}z} z^{k}\left\{1+O(z^{-1})\right\}, \quad (|z| >> \kappa^2). \tag{16}$$

Hence for large κ,

$$\begin{aligned} W_{k,m}(z) = (2\pi)^{-\frac{1}{2}}\left\{(k+m-\tfrac{1}{2})!(k-m-\tfrac{1}{2})!\right\}^{\frac{1}{2}}\left(1-\frac{4k}{z}+\frac{4m^2}{z^2}\right)^{-\frac{1}{4}} \\ \times \exp\left\{-\tfrac{1}{2}z\left(1-\frac{4k}{z}+\frac{4m^2}{z^2}\right)^{\frac{1}{2}} + k\cosh^{-1}\left(\frac{z-2k}{2\kappa}\right)\right. \\ \left. -|m|\cosh^{-1}\left(\frac{kz-2m^2}{\kappa z}\right)\right\}\sum_{i=0}^{\infty} v_i. \end{aligned} \tag{17}$$

The second solution to the original differential equation follows from (2), or alternatively directly from (17). By either route,

$$W_{-k,m}\left(ze^{\pm i\pi}\right) = e^{\mp i\pi k}(2\pi)^{\frac{1}{2}}\left\{\left(k+m-\tfrac{1}{2}\right)!\left(k-m-\tfrac{1}{2}\right)!\right\}^{-\frac{1}{2}}$$

$$\times\left(1-\frac{4k}{z}+\frac{4m^2}{z^2}\right)^{-\frac{1}{4}} \exp\left\{\frac{1}{2}z\left(1-\frac{4k}{z}+\frac{4m^2}{z^2}\right)^{\frac{1}{2}}\right.$$

$$\left. - k\cosh^{-1}\left(\frac{z-2k}{2k}\right) + |m|\cosh^{-1}\left(\frac{kz-2m^2}{kz}\right)\right\}\sum_{i=0}^{\infty}(-1)^i v_i . \quad (18)$$

For large m , the zero-order approximations in (17) and (18) reduce to results derived by Kazarinoff (1955) by application of Langer's methods (1932, 1949).

When $\arg k = \frac{1}{2}\pi$, it is more appropriate to re-express the series for v by separating y into its real and imaginary parts, thus

$$y = \frac{\frac{1}{2}z - k}{\left\{\left(\frac{1}{2}z-k\right)^2+\mu^2\right\}^{\frac{1}{2}}} + i\,\frac{\mu}{\left\{\left(\frac{1}{2}z-k\right)^2+\mu^2\right\}^{\frac{1}{2}}} , \qquad (19)$$

with the abbreviations

$$\left.\begin{array}{c} \mu = \left(m^2-k^2\right)^{\frac{1}{2}},\ A = 1-i\beta,\ a = 1+i\beta,\ B = 1-3i\beta,\ b = 1+3i\beta, \\ \beta = k/\mu . \end{array}\right\} \quad (20)$$

If in addition the further new variable θ is introduced by

$$y = e^{i\theta}, \tag{21}$$

so that

$$\theta = \tan^{-1}\left\{\frac{2\mu}{x-2k}\right\}, \tag{22}$$

recurrence relation (9) becomes

$$2\mu v_{i+1} = \sin^2\theta\left(\cos\theta - \beta\sin\theta\right)\frac{dv_i}{d\theta} + \frac{1}{4}\int^{\theta}\left\{\cos\theta\left(5\sin^2\theta-1\right)+\beta\sin\theta\left(5\cos^2\theta-2\right)\right\}v_i\,d\theta, \tag{23}$$

from which successive v_i's may be derived. Alternatively, these can be deduced by direct substitution of (20) and (21) into (13). Either approach yields

$$v_i(i \text{ odd}) = \frac{\gamma_i}{\mu^i}\sum_{r=0}^{\frac{1}{2}(3i-1)}\left\{c_{ri}\sin(3i-2r)\theta + \beta d_{ri}\cos(3i-2r)\theta\right\}, \tag{24a}$$

$$V_i(i\,\text{even}) = \frac{\gamma_i}{\mu^i} \sum_{r=0}^{\frac{3}{2}i} \left\{ c_{ri} \cos(3i-2r)\theta + \beta d_{ri} \sin(3i-2r)\theta \right\}. \tag{24b}$$

The first few coefficients in (24a) and (24b) are

$$\gamma_1 = -\frac{1}{96}; \quad c_{01} = 5,\ c_{11} = -3,\ d_{01} = -5,\ d_{11} = 9;$$

$$\gamma_2 = \frac{-1}{36\,864}; \quad c_{02} = 385(1-\beta^2),\ c_{12} = -462(1-3\beta^2),$$

$$c_{22} = -3(107+645\beta^2),\ c_{32} = 2(199+451\beta^2),$$

$$d_{02} = 770,\ d_{12} = -1848,\ d_{22} = 1290;$$

$$\gamma_3 = \frac{1}{106\,168\,320}; \quad c_{03} = 425\,425(1-3\beta^2),$$

$$c_{13} = -765\,765(1-7\beta^2),$$

$$c_{23} = -(212\,472 + 8\,454\,420\beta^2),$$

$$c_{33} = 964\,320 + 5\,812\,980\beta^2,$$

$$c_{43} = -(299\,070 + 1\,214\,370\,\beta^2),$$

$$d_{03} = -425\,425(3-\beta^2),\ d_{13} = 765\,765(5-3\beta^2),$$

$$d_{23} = -(3\,169\,296 - 5\,072\,652\,\beta^2),$$

$$d_{33} = -(721\,320 + 5\,812\,980\,\beta^2),$$

$$d_{43} = 2\,112\,210 + 3\,643\,110\,\beta^2\ ;$$

$$\gamma_4 = \frac{1}{81\,537\,269\,760}\ ;$$

$$c_{04} = 185\,910\,725\,(1+\beta^4) - 1\,115\,464\,350\,\beta^2,$$

$$c_{14} = -(446\,185\,740 - 5\,354\,228\,880\,\beta^2 + 1\,338\,557\,220\,\beta^4),$$

$$c_{24} = 21\,437\,130 - 9\,669\,205\,260\,\beta^2 + 4\,134\,757\,770\,\beta^4,$$

$$c_{34} = 646\,789\,924 + 7\,201\,622\,736\,\beta^2 - 7\,095\,439\,252\,\beta^4,$$

$$c_{44} = -(354\,776\,319 + 244\,213\,614\beta^2 - 7\,228\,435\,113\beta^4),$$

$$c_{54} = -(191\,248\,920 + 2\,960\,004\,960\beta^2 + 4\,641\,216\,840\beta^4),$$

$$c_{64} = 138\,073\,200 + 1\,408\,263\,960\beta^2 + 1\,493\,089\,800\beta^4,$$

$$d_{04} = 743\,642\,900(1-\beta^2),$$

$$d_{14} = -(2\,677\,114\,440 - 4\,461\,857\,400\beta^2),$$

$$d_{24} = 2\,799\,379\,440 - 11\,026\,020\,720\beta^2,$$

$$d_{34} = 540\,606\,440 + 14\,190\,878\,504\beta^2,$$

$$d_{44} = -(2\,591\,483\,076 + 9\,637\,913\,484\beta^2),$$

$$d_{54} = 1\,433\,968\,560 + 3\,094\,144\,560\beta^2. \qquad (25)$$

It is noted that the corresponding terms for the modified Bessel function $K_p(z)$ as given in § 4.2, are recovered on setting $k = 0$, $m = p$ and $q = \sin\theta$ in (24a,b).

Ranges of applicability.

Even on the real axis ($\arg \kappa = \arg(z-2k)$), it is necessary to consider four cases:

(i) For $1 < (\frac{1}{2}z-k)/\kappa < \infty$, y is always real and (17) and (18) can be applied as they stand.

(ii) For $-1 < (\frac{1}{2}z-k)/\kappa < 1$, y is imaginary and it is useful to change the variable to

$$y' = \left\{ \frac{z-2k+2\kappa}{-z+2k+2\kappa} \right\}^{\frac{1}{2}} .$$

Analytic continuation then yields expansions applicable in this range.

(iii) Expansions applicable when $-\infty < (\frac{1}{2}z-k)/\kappa < -1$ are most easily deduced from (17) and (18) by reversing the sign of κ .

(iv) The results of this section are inapplicable in the transitional region $\frac{1}{2}z - k \sim k$; treatment of this case will be deferred until § 6.4.

6.3 Transitional and uniform expansions for $W_{k,m}$; k large, m small.

The derivation of transitional and uniform expansions for Whittaker functions with k large involves substantially the same techniques as those exemplified in sections 4 and 5 for modified Bessel functions and parabolic cylinder functions. Consequently only the most important equations and results will be recorded here.

Let $M(\mu)$, the Mellin transform of V, be defined as

$$V = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M(\mu)\, x^{-\mu}\, d\mu, \tag{1}$$

where now $x = 3t^3/8k$. Then if we write

$$M(\mu) = (\mu-1)!\left(-\mu-\tfrac{5}{6}\right)!\left(-\mu-\tfrac{1}{6}\right)!\; m(\mu), \tag{2}$$

substitution for V into (6.1.11) leads to the recurrence relation

$$m(\mu) = -2\left(\frac{3}{8k}\right)^{\frac{2}{3}} \int \left(\mu+\tfrac{1}{6}\right) \frac{\left(\mu-\frac{2}{3}\right)!\left(-\mu-\frac{1}{2}\right)!}{\left(-\mu-\frac{5}{6}\right)!\ \mu!}\, m\left(\mu+\tfrac{1}{3}\right)$$

$$+\left(\frac{3}{8k}\right)^{\frac{4}{3}} \int \left(\mu-\tfrac{1}{6}\right)\left(\mu^2-\tfrac{1}{3}\mu-\frac{M}{36}\right)\frac{\left(\mu-\frac{4}{3}\right)!\left(-\mu-\frac{1}{2}\right)!}{\left(-\mu-\frac{1}{6}\right)!\ \mu!}\, m\left(\mu-\tfrac{1}{3}\right). \tag{3}$$

This summation problem may be solved for $m(\mu)$ by successive approximations if we write

$$m(\mu) = \sum_{i=0}^{\infty} m_i(\mu) \tag{4}$$

The $(i+1)$ th contribution to $m(\mu)$ is then given in terms of the i th and $(i-1)$ th contributions by

$$m_{i+1}(\mu) = -2\left(\frac{3}{8k}\right)^{\frac{2}{3}} \int (\mu+\tfrac{1}{6}) \frac{(\mu-\frac{2}{3})!\,(-\mu-\frac{1}{2})!}{(-\mu-\frac{5}{6})!\;\mu!}\, m_i(\mu+\tfrac{1}{3})$$

$$+\left(\frac{3}{8k}\right)^{\frac{4}{3}} \int (\mu-\tfrac{1}{6})\left(\mu^2-\tfrac{1}{3}\mu-\frac{\eta}{36}\right)\frac{(\mu-\frac{4}{3})!\,(-\mu-\frac{1}{2})!}{(-\mu-\frac{1}{6})!\;\mu!}\, m_{i-1}(\mu-\tfrac{1}{3}); \tag{5}$$

here contributions are ordered according to their power in $k^{-2/3}$. Successive summation, with $m_0(\mu)=\frac{1}{2\pi}$, yields

$$2\pi M(\mu) = (\mu-1)!\,(-\mu-\tfrac{5}{6})!\,(-\mu-\tfrac{1}{6})! + \frac{6}{5}\left(\frac{3}{8k}\right)^{\frac{2}{3}}(\mu-\tfrac{2}{3})!\,(-\mu+\tfrac{1}{2})!\,(-\mu-\tfrac{1}{6})!$$

$$+\frac{18}{25}\left(\frac{3}{8k}\right)^{\frac{4}{3}}\left(\mu^3-\frac{79}{42}\mu^2+\frac{233}{252}\mu-\frac{103}{756}+\frac{25\eta}{216}\right)(\mu-\tfrac{4}{3})!\,(-\mu+\tfrac{1}{2})!\,(-\mu-\tfrac{5}{6})!$$

$$+\frac{36}{125}\left(\frac{3}{8k}\right)^{2}\left\{\mu^5-\frac{65}{14}\mu^4+\frac{2935}{378}\mu^3-\left(\frac{2915}{504}-\frac{25}{72}\eta\right)\mu^2\right.$$

$$\left.+\left(\frac{16823}{9072}-\frac{25}{54}\eta\right)\mu+\frac{125}{2592}(3\eta-4)\right\}(\mu-1)!\,(-\mu-\tfrac{5}{6})!\,(-\mu-\tfrac{1}{6})!$$

$$+\cdots \tag{6}$$

The constant in the last term of (6) has been chosen so as to achieve agreement with the Green-type expansions of § 6.1.

6.31 Transitional expansions.

An expansion for V valid in the transitional region $z \sim 4k$ follows from (6.3.1) and (6.3.6) if the contour, chosen with the usual precautions+ regarding the separation of the poles of $M(\mu)$, is closed in a clockwise direction. The result is

$$V = \left(-\tfrac{2}{3}\right)! \left(\tfrac{2}{3}k\right)^{\frac{1}{6}} (\pi t)^{-\frac{1}{2}} \Bigg[1 + \frac{\left(-\frac{1}{6}\right)!}{840\,\pi^{\frac{1}{2}}} \left(\frac{3}{4k}\right)^{\frac{4}{3}} (9 - 35\eta) - \frac{7 - 45\eta}{14400 k^2} \cdots$$

$$- \frac{3\left(-\frac{1}{6}\right)!}{2t^2\,\pi^{\frac{1}{2}}} \left(\frac{4}{3}k\right)^{\frac{2}{3}} \left\{ 1 - \frac{2\left(\frac{1}{6}\right)!}{5\pi^{\frac{1}{2}}} \left(\frac{3}{4k}\right)^{\frac{2}{3}} - \frac{1}{3150 k^2} \cdots \right\}$$

$$+ \frac{4k}{3t^3} \left\{ 1 + \frac{\left(-\frac{1}{6}\right)!}{840\,\pi^{\frac{1}{2}}} \left(\frac{3}{4k}\right)^{\frac{4}{3}} (9 - 35\eta) - \frac{7 - 45\eta}{14400 k^2} \cdots \right\}$$

$$- \frac{3\left(-\frac{1}{6}\right)!}{2t^4\,\pi^{\frac{1}{2}}} \left(\frac{4}{3}k\right)^{\frac{2}{3}} \left\{ 1 - \frac{\left(\frac{1}{6}\right)!}{80\,\pi^{\frac{1}{2}}} \left(\frac{3}{4k}\right)^{\frac{2}{3}} (17 - 5\eta) + \cdots \right\} \cdots \Bigg]. \quad (1)$$

+ This involves loops around $\mu = 1/6$ and $\mu = 1/3$, the poles nearest the origin of the third term in (6.3.6).

The corresponding series for $W_{-k,m}\left(z e^{\pm i\pi}\right)$ is obtained on replacing k by $k e^{\pm i\pi}$.

6.32 Expansions at the transition point.

Equation (6.31.1) shows that at the transition point $z = 4k$, where $t \to \infty$, $W_{k,m}(z)$ and $W_{-k,m}(-z)$ are asymptotically represented by

$$W_{k,m}(4k) = \frac{1}{\pi}\left\{\left(k+m-\tfrac{1}{2}\right)!\left(k-m-\tfrac{1}{2}\right)!\right\}^{\frac{1}{2}}\left(-\tfrac{2}{3}\right)!\left(\frac{k}{12}\right)^{\frac{1}{6}}$$

$$\times\left\{1+\frac{\left(-\frac{1}{6}\right)!}{840\pi^{\frac{1}{2}}}\left(\frac{3}{4k}\right)^{\frac{4}{3}}(9-35\eta)-\frac{7-45\eta}{14400k^2}\cdots\right\}, \quad (1)$$

$$W_{-k,m}\left(4ke^{\pm i\pi}\right) = \frac{e^{\mp i\pi\left(k-\frac{1}{6}\right)}}{\left\{\left(k+m-\frac{1}{2}\right)!\left(k-m-\frac{1}{2}\right)!\right\}^{1/2}}\left(-\tfrac{2}{3}\right)!\left(\frac{16k}{3}\right)^{\frac{1}{6}}$$

$$\times\left\{1-\frac{e^{\pm\frac{2}{3}i\pi}}{840\pi^{\frac{1}{2}}}\left(-\tfrac{1}{6}\right)!\left(\frac{3}{4k}\right)^{\frac{4}{3}}(9-35\eta)-\frac{(7-45\eta)}{14400k^2}\cdots\right\}. \quad (2)$$

6.33 Uniform expansions.

Defining the subsidiary function $k(x)$ by

$$k(x) = \frac{1}{4\pi^2 i}\int_{c-i\infty}^{c+i\infty}(\mu-1)!\left(-\mu-\tfrac{5}{6}\right)!\left(-\mu-\tfrac{1}{6}\right)!\; x^{-\mu}\,d\mu, \quad (1)$$

where $x = 3t^3/8k$, we obtain by analogy with §§ 4.5 and 5.6 the following uniform expansion for

$$v = k(x)\left\{1 + \frac{t}{k}\sum_{n=1}^{\infty}(-k)^n g_n(x)t^{-5n}\right\}$$
$$- \frac{9}{20}\frac{t}{k}k'(x)\sum_{n=0}^{\infty}(-k)^n h_n(x)t^{-5n}. \qquad (2)$$

Rather lengthy calculation yields for the first few coefficients

$$\left.\begin{aligned} g_1(x) &= \frac{1}{1260}\left\{(9-35\eta)(x^2+3x)-126\right\}, \\ g_2(x) &= \frac{2}{42525}\left\{(213-1575\eta)x^3-192x^2+7614x+2268\right\}, \end{aligned}\right\} \qquad (3a)$$

$$\left.\begin{aligned} h_0(x) &= 1, \\ h_1(x) &= \frac{2}{945}\left\{(45-175\eta)x^3-1350x-756\right\}, \\ h_2(x) &= \frac{16}{127575}\left\{(85-1575\eta)x^4-(1215-4725\eta)x^3 \right. \\ &\qquad \left. + 61110x^2+72900x+13608\right\}. \end{aligned}\right\} \qquad (3b)$$

The Liouville-Green factors completing the expansions for $W_{k,m}(z)$ and $W_{-k,m}(-z)$ are as quoted in

(6.1.21) and (6.1.22).

It may be shown that the zero-order approximation in (2) is in agreement with results given previously in terms of Airy functions by Erdélyi and Swanson (1957), and Slater (1960, § 4.6.4); and in terms of $K_{\frac{1}{3}}$ by Taylor (1939).

6.4 Transitional and uniform expansions for $W_{k,m}$; $|k^2 - m^2|$ large.+

The Mellin transform of v is defined by (6.3.1); but now with $x = 3Ay^3/16K$, where A and K are as in (6.2.8). With $m(\mu)$ defined by (6.3.2) and also $m(\mu) = \sum_{i=0}^{\infty} m_i(\mu)$, (6.2.7) yields the recurrence relation

$$m_{i+1}(\mu) = -\frac{B}{A}\left(\frac{3A}{16K}\right)^{\frac{2}{3}} \int \left(\mu + \tfrac{1}{6}\right) \frac{(\mu - \frac{2}{3})!\,(-\mu - \frac{1}{2})!}{(-\mu - \frac{5}{6})!\ \mu!}\, m_i\left(\mu + \tfrac{1}{3}\right)$$

$$-\frac{b}{A}\left(\frac{3A}{16K}\right)^{\frac{4}{3}} \int \left(\mu - \tfrac{1}{6}\right)^3 \frac{(\mu - \frac{4}{3})!\,(-\mu - \frac{1}{2})!}{(-\mu - \frac{1}{6})!\ \mu!}\, m_{i-1}\left(\mu - \tfrac{1}{3}\right)$$

$$-\frac{a}{A}\left(\frac{3A}{16K}\right)^{2} \int \frac{(\mu - \frac{5}{6})^2 (\mu - \frac{1}{6})^2}{\mu(\mu - 1)}\, m_{i-2}(\mu - 1). \qquad (1)$$

+ The case k and m large will be treated in Kummer's notation in chapter 7 (see also Kazarinoff 1957).

Here, the constants A , B , a and b are as in (6.2.8) and contributions are ordered according to their power in $k^{-2/3}$.

Summations arising from the last term in (1) are of a different type to those encountered earlier. Their evaluation will be illustrated by the simplest example occurring, that for m_{i-2} = constant. Writing

$$\frac{(\mu-\frac{5}{6})^2(\mu-\frac{1}{6})^2}{\mu(\mu-1)} = \mu^2-\mu-\frac{5}{18}+\frac{25}{1296\mu(\mu-1)}, \tag{2}$$

we can sum the first part by either of the methods indicated for (4.3.20), and the second by noting that

$$\int \frac{1}{\mu(\mu-1)} = \frac{-1}{\mu-1}. \tag{3}$$

Choosing the constant in m_3 to reach agreement with the Green-type expansion of § 6.2, we find that

$$2\pi M(\mu) = (\mu-1)!\,(-\mu-\tfrac{5}{6})!\,(-\mu-\tfrac{1}{6})!$$

$$+\frac{3}{5}\frac{B}{A}\left(\frac{3A}{16k}\right)^{\frac{2}{3}}(\mu-\tfrac{2}{3})!\,(-\mu+\tfrac{1}{2})!\,(-\mu-\tfrac{1}{6})!$$

$$+ \frac{9}{50}\left(\frac{B}{A}\right)^2\left(\frac{3A}{16K}\right)^{\frac{4}{3}}\left(\mu-\frac{47}{42}\right)\left(\mu+\frac{1}{6}\right)\left(\mu-\frac{1}{3}\right)!\left(-\mu+\frac{1}{2}\right)!\left(-\mu-\frac{5}{6}\right)!$$

$$+ \frac{3}{7}\frac{b}{A}\left(\frac{3A}{16K}\right)^{\frac{4}{3}}\left(\mu^2-\frac{4}{3}\mu+\frac{19}{36}\right)\left(\mu-\frac{4}{3}\right)!\left(-\mu+\frac{1}{2}\right)!\left(-\mu-\frac{5}{6}\right)!$$

$$\left(\frac{3A}{16K}\right)^2\left[\frac{9}{250}\left(\frac{B}{A}\right)^3+\left\{-\frac{18}{175}\left(\frac{B}{A}\right)^3+\frac{9}{35}\frac{bB}{A^2}\right\}\mu^4\right.$$

$$+\left\{\frac{127}{2100}\left(\frac{B}{A}\right)^3-\frac{92}{105}\frac{bB}{A^2}-\frac{1}{3}\frac{a}{A}\right\}\mu^3$$

$$+\left\{\frac{9}{350}\left(\frac{B}{A}\right)^3+\frac{69}{70}\frac{bB}{A^2}+\frac{a}{A}\right\}\mu^2$$

$$+\left\{-\frac{1459}{84000}\left(\frac{B}{A}\right)^3-\frac{127}{315}\frac{bB}{A^2}-\frac{17}{18}\frac{a}{A}\right\}\mu$$

$$\left.+\frac{1}{648A^2}\left(199-451\sigma^2\right)\right]\left(\mu-1\right)!\left(-\mu-\frac{5}{6}\right)!\left(-\mu-\frac{1}{6}\right)!$$

$$+\left(\frac{3A}{16K}\right)^2\frac{25a}{1296A}\left(\mu-2\right)!\left(-\mu-\frac{5}{6}\right)!\left(-\mu-\frac{1}{6}\right)!+\cdots. \qquad (4)$$

6.41 <u>Transitional expansions.</u>

Separation of the poles of $(\mu-1)!$, etc., from those of $\left(-\mu-\frac{1}{6}\right)!\left(-\mu-\frac{5}{6}\right)!$, etc., and closure of the contour of integration in a clockwise direction, yields the following expansion for v in the transition region $z - 2k \sim 2K$:

$$V = \left(-\tfrac{2}{3}\right)!\left(\frac{4K}{3A}\right)^{\frac{1}{6}}(\pi y)^{-\frac{1}{2}}\Bigg[1 - \frac{\left(-\frac{1}{6}\right)!}{1120\,\pi^{\frac{1}{2}}}\left(\frac{3}{A^2}\right)^{\frac{1}{3}}\frac{B^2+15bA}{K^{4/3}}$$

$$-\frac{1}{921600AK^2}\left(64B^3-160bBA-3375aA^2\right)\cdots$$

$$-\frac{\left(-\frac{1}{6}\right)!}{2y^2\pi^{\frac{1}{2}}}3^{\frac{1}{3}}\left(\frac{8K}{A}\right)^{\frac{2}{3}}\left\{1-\frac{\left(\frac{1}{6}\right)!}{20\pi^{\frac{1}{2}}}\left(\frac{9}{A}\right)^{\frac{1}{3}}\frac{B}{K^{2/3}}\right.$$

$$\left.+\frac{1}{6451200AK^2}\left(736B^3+3200bBA-32375aA^2\right)\cdots\right\}$$

$$+\frac{8K}{3Ay^3}\left\{1-\frac{\left(-\frac{1}{6}\right)!}{1120\,\pi^{\frac{1}{2}}}\left(\frac{3}{A^2}\right)^{\frac{1}{3}}\frac{B^2+15bA}{K^{4/3}}\right.$$

$$\left.-\frac{1}{921600AK^2}\left(64B^3-160bBA-3375aA^2\right)\cdots\right\}$$

$$-\frac{\left(-\frac{1}{6}\right)!}{4y^4\pi^{\frac{1}{2}}}3^{\frac{1}{3}}\frac{B}{A}\left(\frac{8K}{A}\right)^{\frac{2}{3}}\left\{1-\frac{\left(\frac{1}{6}\right)!}{40\pi^{\frac{1}{2}}}\left(\frac{9}{A}\right)^{\frac{1}{3}}\frac{2B^2+5bA}{BK^{2/3}}+\cdots\right\}+\cdots\Bigg]. \quad (1)$$

Corresponding results for u follow on replacing κ and k by $\kappa e^{\pm i\pi}$ and $k e^{\pm i\pi}$, respectively.

6.42 <u>Transition point</u>.

At the transition point $z = 2(k+\kappa)$, where $y \to \infty$, (6.41.1) shows that $W_{k,m}(z)$ and $W_{-k,m}(-z)$ have the asymptotic expansions

$$W_{k,m}(2k+2\kappa) = \frac{1}{2\pi}\left\{(k+m-\tfrac{1}{2})!\,(k-m-\tfrac{1}{2})!\right\}^{\frac{1}{2}}\left(1+\frac{k}{\kappa}\right)^{\frac{1}{2}}$$

$$\times\left(\frac{4\kappa}{3A}\right)^{\frac{1}{6}}(-\tfrac{2}{3})!\left[1-\frac{(-\frac{1}{6})!}{1120\pi^{\frac{1}{2}}}\left(\frac{3}{A^2}\right)^{\frac{1}{3}}\frac{B^2+15bA}{\kappa^{4/3}}\right.$$

$$\left.-\frac{1}{921\,600\,A\kappa^2}\left(64B^3-160bBA-3375aA^2\right)+\cdots\right], \quad (1)$$

$$W_{-k,m}\left\{-2(k+\kappa)e^{\pm i\pi}\right\} = \frac{2\,e^{\mp i\pi(k-\frac{1}{6})}}{\left\{(k+m-\frac{1}{2})!\,(k-m-\frac{1}{2})!\right\}^{\frac{1}{2}}}\left(1+\frac{k}{\kappa}\right)^{\frac{1}{2}}$$

$$\times\left(\frac{4\kappa}{3A}\right)^{\frac{1}{6}}(-\tfrac{2}{3})!\left[1-\frac{(-\frac{1}{6})!}{1120\pi^{\frac{1}{2}}}\left(\frac{3}{A^2}\right)^{\frac{1}{3}}\frac{B^2+15bA}{\kappa^{4/3}}\,e^{\pm\frac{2}{3}i\pi}\right.$$

$$-\frac{1}{921\,600\,A\mathfrak{K}^2}\left(64\,B^3-160\,bBA-3375\,aA^2\right)+\cdots\Bigg]. \qquad (2)$$

6.43 <u>Uniform expansions</u>.

By the method of § 4.5, (6.4.4) yields the following uniform expansion:

$$v = k(x)\left\{1-\frac{y}{\mathfrak{K}}\sum_{n=1}^{\infty}\mathfrak{K}^n\, i_n(x)\, A^{-2n}\, y^{-5n}\right\}$$
$$-\frac{9}{80}\,\frac{y}{\mathfrak{K}}\,k'(x)\sum_{n=0}^{\infty}\mathfrak{K}^n\, j_n(x)\, A^{-2n}\, y^{-5n}. \qquad (1)$$

The first few coefficients in (1) are

$$i_1(x) = \frac{1}{2520}\left\{(8B^2+120\,bA)x^2+(24B^2+360\,bA)x + 63B^2\right\},$$

$$i_2(x) = \frac{1}{42525}\left\{(744\,B^3-600\,bBA-37625\,aA^2)x^3 + (1104\,B^3+4800\,bBA-42000\,aA^2)x^2 + (5832\,B^3+8100\,bBA)\,x + 1134\,B^3\right\}. \qquad (2a)$$

$$j_0(x) = B,$$

$$j_1(x) = \frac{8}{945}\left\{(20B^2 + 300bA)x^3 + 450(B^2 + bA)x + 189B^2\right\},$$

$$j_2(x) = \frac{32}{127575}\left\{(1480B^3 + 2600bBA - 70000aA^2)x^4\right.$$

$$+ (1080B^3 + 16200bBA)x^3$$

$$+ (62055B^3 + 126000bBA - 63000aA^2)x^2$$

$$\left. + 48600(B^3 + bBA)x + 6804B^3\right\}. \qquad (2b)$$

The Liouville-Green factors are as quoted in (6.2.17) and (6.2.18).

Special case of Bessel functions.

Transitional and uniform expansions for modified Bessel functions may be recovered from the foregoing results for Whittaker functions by comparing the respective integro-differential equations for V, that is (4.1.9) and (6.2.7). If it is supposed that A, B, a and b are now independently adjustable parameters --- i.e. ignoring their mutual dependence in terms of γ and k --- (6.2.7) is seen to pass over into (4.1.9) with the substitutions $y = q$, $A = B = 1$; $a = b = 0$ and $k = -\frac{1}{8}p$. Repetition of these substitutions in (6.41.1) and (1) gives the corresponding expansions for modified Bessel functions (4.42.1) and (4.5.13).

6.5 Green-type, transitional and uniform expansions of the Poiseuille functions $Pe(r, \sigma)$ and $Qe(r, \sigma)$ for large σ .

Poiseuille functions are solutions of the differential equation

$$\frac{d^2 y}{d r^2} + \frac{1}{r}\frac{dy}{dr} - 4\sigma^2(1-r^2)\,y = 0 \qquad (1)$$

where $0 \leq r \leq 1$. Brinkman (1951) has shown that this differential equation arises on consideration of the problem of heat generation in a liquid flowing through a capillary according to Poiseuille's law. In the particular case of real σ , the solutions are generally denoted by $Pe(r,\sigma)$ and $Qe(r,\sigma)$. Their definitions in terms of Whittaker functions are given by Lauwerier (1953, § 5) as

$$Pe(r,\sigma) = -2e^{\frac{1}{2}\pi\sigma} \mathcal{I}m\left\{\frac{W_{\frac{1}{2}i\sigma,0}(2i\sigma r^2)}{(\frac{1}{2}i\sigma-\frac{1}{2})!\, r(2i\sigma)^{\frac{1}{2}}}\right\}, \quad (2)$$

$$Qe(r,\sigma) = -\frac{\pi e^{-\frac{1}{2}\pi\sigma}}{1+e^{-\pi\sigma}} \mathcal{R}e\left\{\frac{W_{\frac{1}{2}i\sigma,0}(2i\sigma r^2)}{(\frac{1}{2}i\sigma-\frac{1}{2})!\, r(2i\sigma)^{\frac{1}{2}}}\right\}, \quad (3)$$

where $\mathcal{I}m$ and $\mathcal{R}e$ denote respectively imaginary and real parts. It is thus required to seek expansions of $W_{k,m}$ with $m = 0$ corresponding to (6.1.21) and (6.1.22) but for the range $0 < z/4k < 1$. This is achieved by means of the continuation formula[+] (Slater 1960, § 2.5.2)

$$W_{k,0}(z) = \frac{\{(k-\frac{1}{2})!\}^2}{2\pi i}\left\{e^{i\pi k}\, W_{-k,0}(ze^{i\pi}) - e^{-i\pi k}\, W_{-k,0}(ze^{-i\pi})\right\}. \quad (4)$$

[+] Owing to the change in range from $1 < z/4k < \infty$ to $0 < z/4k < 1$, the desired results cannot be taken directly from (6.1.21) and (6.1.22).

6.51 <u>Green-type expansions.</u>

From (6.1.22) for $0 < z/4k < 1$,

$$W_{-k,0}\left(ze^{\pm i\pi}\right) = e^{\mp i\pi k}\frac{(2\pi)^{\frac{1}{2}}}{\left(k-\frac{1}{2}\right)!}\left(\frac{4k}{z}-1\right)^{-\frac{1}{4}}e^{\pm\frac{1}{4}i\pi}$$

$$\times\ \exp\left\{\mp\frac{i}{2}\left(4kz\right)^{\frac{1}{2}}\left(1-\frac{z}{4k}\right)^{\frac{1}{2}}\pm 2ik\cos^{-1}\left(\frac{z}{4k}\right)^{\frac{1}{2}}\right\}\sum_{i=0}^{\infty}(\mp 1)^{i}v_i(t), \tag{1}$$

with $v_i(t)$ given by (6.1.19). The substitutions

$$z = 2i\sigma r^2,\ k = \tfrac{1}{2}i\sigma,\ t = e^{\pm\frac{1}{2}i\pi}, \tag{2}$$

combined with (6.5.4) and (1) yield

$$\frac{W_{\frac{1}{2}i\sigma,0}\left(2i\sigma r^2\right)}{\left(\frac{1}{2}i\sigma-\frac{1}{2}\right)!\ r\left(2i\sigma\right)^{\frac{1}{2}}} = \frac{1}{2\left(\pi r\sigma\right)^{\frac{1}{2}}\left(1-r^2\right)^{\frac{1}{4}}}$$

$$\times\left[-i\exp\left\{\sigma\left(r\left(1-r^2\right)^{\frac{1}{2}}-\cos^{-1}r\right)\right\}\sum_{i=0}^{\infty}(-1)^{i}v_i(s)\right.$$

$$\left.+\exp\left\{-\sigma\left(r\left(1-r^2\right)^{\frac{1}{2}}-\cos^{-1}r\right)\right\}\sum_{i=0}^{\infty}v_i(s)\right], \tag{3}$$

where $s = r(1-r^2)^{-\frac{1}{2}}$. Therefore, by (6.5.2), (6.5.3) and (3),

$$Pe(r,\sigma) = \frac{(1-r^2)^{-\frac{1}{4}}}{(\pi r\sigma)^{\frac{1}{2}}} \exp\left\{\sigma\left(r(1-r^2)^{\frac{1}{2}} + \sin^{-1} r\right)\right\} \sum_{i=0}^{\infty} (-1)^i v_i(s), \quad (4)$$

$$Qe(r,\sigma) = \frac{-\pi^{\frac{1}{2}}(1-r^2)^{-\frac{1}{4}}}{2(1+e^{-\pi\sigma})(r\sigma)^{\frac{1}{2}}} \exp\left\{-\sigma\left(r(1-r^2)^{\frac{1}{2}} + \sin^{-1} r\right)\right\} \sum_{i=0}^{\infty} v_i(s). \quad (5)$$

With $v_0 = 1$, the series for v_i becomes

$$v_i = \frac{\delta_i}{\sigma^i} \sum_{r=0}^{2i} f_{ri}\, s^{3i-2r}, \quad (6)$$

where the coefficients may be expressed as

$$\delta_i = 2^i \alpha_i, \quad f_{ri} = (-1)^r a_{ri} \,(\eta = -1), \quad (7)^+$$

+ Lauwerier (1953) appears to have made an algebraic error in his calculation of m_2 on p.64 of his paper. The corrected expression is

$$m_2(a=0) = \frac{(\sin\alpha\cos\alpha)^{-3/2}}{24\cos^2\alpha}\left(-4\sin^4\alpha + 12\sin^2\alpha - 3\right).$$

and the coefficients a_{ri} and α_i are as in (6.1.20).

6.52 Transitional expansions.

Transitional expansions for P_e and Q_e (valid when $r \sim 1$) can be deduced from the corresponding expansion for $W_{-k,0}(z e^{\pm i\pi})$, i.e. (6.31.1) with $k e^{\pm i\pi}$ replacing k, by substitutions (6.51.2). With the continuation formula (6.5.4) these yield a transitional expansion for $W_{\frac{1}{2}i\sigma,0}(2i\sigma r^2)$ valid when $r = 1 - \delta$, where δ is arbitrarily small but positive. By (6.5.2) and (6.5.3) we are then led to the following transitional expansions:

$$Pe(r,\sigma) = \frac{(-\frac{2}{3})!\, e^{\frac{1}{2}\pi\sigma}}{\pi r \sigma^{\frac{1}{3}} 3^{\frac{1}{6}}}\left[\exp\left\{\sigma\left(r(1-r^2)^{\frac{1}{2}} - \cos^{-1} r\right)\right\} \Xi(s)\right.$$

$$\left. - \tfrac{1}{2} \exp\left\{-\sigma\left(r(1-r^2)^{\frac{1}{2}} - \cos^{-1} r\right)\right\} \Xi(s)\right], \qquad (1)$$

where

$$\Xi(s) = 1 + \frac{11\left(-\frac{1}{6}\right)!}{210\,\pi^{\frac{1}{2}}}\left(\frac{3}{2\sigma}\right)^{\frac{4}{3}} + \frac{13}{900\,\sigma^2} + \cdots$$

$$- \frac{3\left(-\frac{1}{6}\right)!}{2s^2\pi^{\frac{1}{2}}}\left(\frac{2\sigma}{3}\right)^{\frac{2}{3}}\left\{1 + \frac{2\left(\frac{1}{6}\right)!}{5\pi^{\frac{1}{2}}}\left(\frac{3}{2\sigma}\right)^{\frac{2}{3}} + \frac{2}{1575\sigma^2} + \cdots\right\}$$

$$+ \frac{2\sigma}{3s^3}\left\{1 + \frac{11\left(-\frac{1}{6}\right)!}{210\pi^{\frac{1}{2}}}\left(\frac{3}{2\sigma}\right)^{\frac{4}{3}} + \frac{13}{900\sigma^2} + \cdots\right\}$$

$$+ \frac{3\left(-\frac{1}{6}\right)!}{2s^4\pi^{\frac{1}{2}}}\left(\frac{2\sigma}{3}\right)^{\frac{2}{3}}\left\{1 + \frac{3\left(\frac{1}{6}\right)!}{20\pi^{\frac{1}{2}}}\left(\frac{3}{2\sigma}\right)^{\frac{2}{3}} + \cdots\right\}$$

$$+ \cdots ; \tag{2}$$

$$Qe(r,\sigma) = -\frac{\left(-\frac{2}{3}\right)!\, 3^{\frac{1}{3}}}{4r\sigma^{\frac{1}{3}}\cosh\left(\frac{1}{2}\pi\sigma\right)}\exp\left\{-\sigma\left(r\left(1-r^2\right)^{\frac{1}{2}} - \cos^{-1} r\right)\right\}$$

$$\times \left[1 - \frac{11\left(-\frac{1}{6}\right)!}{210\pi^{\frac{1}{2}}}\left(\frac{3}{2\sigma}\right)^{\frac{4}{3}} + \frac{13}{900\sigma^2} + \cdots\right.$$

$$+ \frac{3\left(-\frac{1}{6}\right)!}{2s^2\pi^{\frac{1}{2}}}\left(\frac{2\sigma}{3}\right)^{\frac{2}{3}}\left\{1 - \frac{2\left(\frac{1}{6}\right)!}{5\pi^{\frac{1}{2}}}\left(\frac{3}{2\sigma}\right)^{\frac{2}{3}} + \frac{2}{1575\sigma^2} + \cdots\right\}$$

$$- \frac{2\sigma}{3s^3}\left\{1 - \frac{11\left(-\frac{1}{6}\right)!}{210\pi^{\frac{1}{2}}}\left(\frac{3}{2\sigma}\right)^{\frac{4}{3}} + \frac{13}{900\sigma^2} + \cdots\right\}$$

$$-\frac{3(-\frac{1}{6})!}{2\varsigma^4\pi^{\frac{1}{2}}}\left(\frac{2\sigma}{3}\right)^{\frac{2}{3}}\left\{1-\frac{3(\frac{1}{6})!}{20\pi^{\frac{1}{2}}}\left(\frac{3}{2\sigma}\right)^{\frac{2}{3}}+\cdots\right\}+\cdots\Bigg]. \quad (3)$$

6.53 Transition point.

In the limit $\delta \to 0$, i.e. at the transition point $r = 1$, $\varsigma \to \infty$, and (6.52.1) and (6.52.3) reduce to

$$Pe(1,\sigma) = \frac{(-\frac{2}{3})!\, e^{\frac{1}{2}\pi\sigma}}{2\pi\sigma^{\frac{1}{3}}3^{\frac{1}{6}}}\left\{1+\frac{11(-\frac{1}{6})!}{210\pi^{\frac{1}{2}}}\left(\frac{3}{2\sigma}\right)^{\frac{4}{3}}+\frac{13}{900\sigma^2}+\cdots\right\}, \quad (1)$$

$$Qe(1,\sigma) = -\frac{(-\frac{2}{3})!\,3^{\frac{1}{3}}}{4\sigma^{\frac{1}{3}}\cosh(\frac{1}{2}\pi\sigma)}\left\{1-\frac{11(-\frac{1}{6})!}{210\pi^{\frac{1}{2}}}\left(\frac{3}{2\sigma}\right)^{\frac{4}{3}}+\frac{13}{900\sigma^2}+\cdots\right\}. \quad (2)$$

6.54 Uniform expansions.

The uniform expansion for $W_{\frac{1}{2}i\sigma,0}(2i\sigma r^2)$ in the range $0 \leq r \leq 1$ is furnished by that for $W_{-k,0}(z e^{\pm i\pi})$, i.e. (6.32.2) with k replaced by $k e^{\pm i\pi}$, together with the continuation formula (6.5.4) and the substitutions (6.51.2). The Liouville-Green factor can be written down from (6.1.21). Combining the resulting expansion with the definitions (6.5.2) and (6.5.3), we obtain the following uniform expansions for Pe and Qe

$$Pe(r,\sigma) = \frac{e^{\frac{1}{2}\pi\sigma}}{(\pi r\sigma)^{\frac{1}{2}}(1-r^2)^{\frac{1}{4}}} \mathcal{I}m\left\{ i e^{\zeta} v(s,-\sigma) - e^{-\zeta} v(s,\sigma) \right\}, \quad (1)$$

$$Qe(r,\sigma) = -\frac{\pi^{\frac{1}{2}}(1-r^2)^{-\frac{1}{4}}}{4(r\sigma)^{\frac{1}{2}}\cosh(\frac{1}{2}\pi\sigma)} \mathcal{R}e\left\{ e^{-\zeta} v(s,\sigma) \right\}. \quad (2)$$

Here

$$\zeta = \sigma\left\{ r(1-r^2)^{\frac{1}{2}} - \cos^{-1} r \right\}, \quad (3)$$

and

$$v(s,\sigma) = k(x) S_1(s,\sigma) + \frac{9s}{100\sigma} k'(x) S_2(s,\sigma), \quad (4)$$

with

$$S_1(s,\sigma) = 1 + \frac{s}{\sigma} \sum_{n=1}^{\infty} \sigma^n p_n(x) s^{-5n}$$

and

$$S_2(s,\sigma) = \sum_{n=0}^{\infty} \sigma^n l_n(x) s^{-5n}. \quad (5)$$

The first few coefficients are

$$\left.\begin{aligned} P_1(x) &= \frac{1}{630}\left(22x^2 + 66x - 63\right), \\ P_2(x) &= \frac{2}{14175}\left(298x^3 - 32x^2 + 1269x + 378\right), \end{aligned}\right\} \quad (6a)$$

$$\left.\begin{aligned} \ell_0(x) &= 1 \\ \ell_1(x) &= \frac{2}{945}\left(110x^3 - 675x - 378\right), \\ \ell_2(x) &= \frac{8}{127575}\left(830x^4 - 2970x^3 + 30555x^2 + 36450x + 6804\right), \end{aligned}\right\} \quad (6b)$$

with $x = (3s^3/4\sigma)\, e^{-i\pi}$. In the derivation of (1) use has been made of the fact that $V(s, -\sigma)$ is real, as is clear from the integral representations for $k(x)$ and $k'(x)$ given in § 4.5. In the next section, these integral representations are used to derive explicit uniform expansions for P_e and Q_e involving $k(z)$, $k'(z)$, where $z = 3s^3/4\sigma$, and Mellin-Barnes integrals. From a computational standpoint it is, however, more advantageous to express

P_e and Q_e in terms of tabulated functions. This can be achieved by combining a continuation formula given by Watson (1944, p.80, eqn.18) with the well-known relation

$$I_p'(x) = -\left(\frac{p}{x}\right) I_p(x) + I_{p-1}(x). \qquad (7)$$

The preceding results can then be re-expressed as follows:

$$P_e(r,\sigma) = \frac{e^{\frac{1}{2}\pi\sigma}}{(\pi r\sigma)^{\frac{1}{2}}(1-r^2)^{1/4}} \Bigg[e^{\varsigma} \left\{ k(z) S_1(s,-\sigma) + \frac{9s}{10\sigma} k'(z) S_2(s,-\sigma) \right\}$$

$$- \frac{1}{2} e^{-\varsigma - 1/z} \left\{ k(z) S_1(s,\sigma) + \frac{9s}{10\sigma}\left(k(z)/z^2 + k'(z)\right) S_2(s,\sigma) \right\} \Bigg], \qquad (8)$$

$$Q_e(r,\sigma) = -\frac{\pi^{\frac{1}{2}}(1-r^2)^{-\frac{1}{4}} e^{-\varsigma}}{4(3r\sigma)^{\frac{1}{2}}\cosh\left(\frac{1}{2}\pi\sigma\right)} \Bigg[\left\{ e^{-\frac{1}{z}} k(z) + \frac{4}{3}\left(\frac{\pi}{z}^3\right)^{\frac{1}{2}} e^{-\frac{1}{2z}} I_{\frac{1}{3}}\left(\frac{1}{2z}\right) \right\} S_1(s,\sigma)$$

$$+ \frac{9s}{10\sigma} \Bigg\{ e^{-\frac{1}{z}} \left(\frac{k(z)}{z^2} + k'(z) \right) + \frac{2}{3}\left(\frac{\pi^3}{z^5}\right)^{\frac{1}{2}} e^{-\frac{1}{2z}}$$

$$\times \left(\left(1 - \frac{z}{3}\right) I_{\frac{1}{3}}\left(\frac{1}{2z}\right) - I_{-\frac{2}{3}}\left(\frac{1}{2z}\right) \right) \Bigg\} S_2(s,\sigma) \Bigg]. \qquad (9)$$

The functions $e^{-1/2z} I_{\frac{1}{3}}(1/2z)$ and $e^{-1/2z} I_{-\frac{2}{3}}(1/2z)$ are extensively tabulated (see, for example, N.B.S. 1949).

6.55 Mellin-Barnes integrals for Poiseuille functions.

Substituting the Mellin-Barnes integral representations for $k(x)$ and $k'(x)$ into (6.54.1) and (6.54.2) and then separating real and imaginary parts, it is found that the uniform expansions for Pe and Qe can be written in the forms

$$Pe(r,\sigma) = \frac{e^{\frac{1}{2}\pi\sigma}}{(\pi r\sigma)^{\frac{1}{2}}(1-r^2)^{\frac{1}{4}}}$$

$$\times \left[e^{\zeta} \left\{ S_1(s,-\sigma)k(z) + \frac{9s}{10\sigma} S_2(s,-\sigma)k'(z) \right\} \right.$$

$$+ \frac{i}{2} e^{-\zeta} \left\{ S_1(s,\sigma) \int_{c-i\infty}^{c+i\infty} \frac{(-\mu-\frac{5}{6})!\,(-\mu-\frac{1}{6})!}{(-\mu)!} z^{-\mu} d\mu \right.$$

$$\left.\left. + \frac{9s}{10\sigma} S_2(s,\sigma) \int_{c-i\infty}^{c+i\infty} \frac{(-\mu+\frac{5}{6})!\,(-\mu+\frac{1}{6})!}{(-\mu)!} z^{-\mu} d\mu \right\} \right], \quad (1)$$

$$Q_e(r,\sigma) = \frac{-e^{-\zeta}}{4\pi i(1-r^2)^{\frac{1}{4}}\cosh(\frac{1}{2}\pi\sigma)}\left(\frac{\pi}{r\sigma}\right)^{\frac{1}{2}}$$

$$\times \left\{ S_1(s,\sigma)\int_{c-i\infty}^{c+i\infty}(\mu-1)!\,(-\mu-\tfrac{5}{6})!\,(-\mu-\tfrac{1}{6})!\,\cos(\pi\mu)\,z^{-\mu}\,d\mu \right.$$

$$\left. + \frac{9s}{10\sigma}S_2(s,\sigma)\int_{c-i\infty}^{c+i\infty}(\mu-1)!\,(-\mu-\tfrac{5}{6})!\,(-\mu-\tfrac{1}{6})!\,\cos(\pi\mu)\,z^{-\mu}\,d\mu \right\}. \quad (2)$$

Here $z = 3s^3/4\sigma$, and S_1 and S_2 are as defined in (6.54.5). It will be observed that the exponentially decreasing term in (1) contributes only to the transitional expansion; this accounts for the apparent discrepancy in form between (6.52.1) and (6.51.4).

Appendix A. Expansion for $W_{k,m}(z)$ when $|z| >> |k|$.

The following integral representation for $\left\{W_{k,m}(z)\right\}^2$ is given in Buchholz (1953, p.85)

$$\left\{W_{k,m}(z)\right\}^2 = \frac{2z}{(-k+m-\frac{1}{2})!\,(-k-m-\frac{1}{2})!}$$

$$\times \int_0^\infty e^{-z\cosh v} K_{2m}(z\sinh v)(\coth\tfrac{1}{2}v)^{2k}\,dv. \qquad \text{(A1)}$$

This representation is valid for $\mathrm{Re}\,(-k \pm m + \frac{1}{2}) > 0$ and $|\arg z| \leq \frac{1}{2}\pi$, K_{2m} being the modified Bessel function as defined in § 4.1.

The first few terms of the series for $W_{k,m}(z)$, in the limit $|z| >> |k|$, can be obtained by replacing K_{2m} by its Green-type expansion. From §§ 4.1 and 4.2 it is seen that for large z and $|\arg z| < \frac{3\pi}{2}$,

$$K_{2m}(z\sinh v) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} e^{-z\sinh v}(z\sinh v)^{-\frac{1}{2}}\left[1 + \frac{\eta}{8z\sinh v} + O(z^{-2})\right], \qquad \text{(A2)}$$

where $\eta = 16m^2 - 1$. Substituting (A2) into (A1), it follows that for $|\arg z| \leq \frac{1}{2}\pi$ the saddle-point lies outside the range of integration. It is therefore permissible to integrate term by term, taking the main contributions as arising only from small values of v. Thus, expanding the integrand about the point $v = 0$, we derive that

$$\left\{ W_{k,m}(z) \right\}^2 \sim \frac{2^{2k} (2\pi z)^{\frac{1}{2}} e^{-z}}{(-k+m-\frac{1}{2})!\,(-k-m-\frac{1}{2})!}$$

$$\times \int_0^\infty e^{-2v} v^{-2k-\frac{1}{2}} \left(1 + \frac{\eta}{8zv} + \dots\right) dv. \qquad \text{(A3)}$$

Completing the integrations and then expanding the resulting factorial functions by Stirling's formula, it follows that for large k

$$\left\{ W_{k,m}(z) \right\}^2 \sim \frac{2\pi e^{-z} z^{2k} e^{2k} k^{-2k}}{(-k+m-\frac{1}{2})!\,(-k-m-\frac{1}{2})!} \left(1 - \frac{3\eta - 1}{48k} + \dots\right). \qquad \text{(A4)}$$

Hence for $|z| >> |k|$ and k negative,

$$W_{k,m}(z) \sim \frac{(2\pi)^{\frac{1}{2}} e^{-\frac{1}{2}z} z^{k} e^{k} k^{-k}}{\left\{(-k+m-\frac{1}{2})!\,(-k-m-\frac{1}{2})!\right\}^{\frac{1}{2}}}\left(1-\frac{3\eta-1}{96k}+\cdots\right). \quad \text{(A5)}$$

Furthermore, by a known continuation formula (Slater 1960, (2.5.19)), (A5) becomes

$$W_{k,m}(z) \sim \frac{\left\{(k+m-\frac{1}{2})!\,(k-m-\frac{1}{2})!\right\}^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}} e^{\frac{1}{2}z} z^{-k} e^{k} k^{-k}}\left(1-\frac{3\eta-1}{96k}+\cdots\right), \quad \text{(A6)}$$

for positive k .

7. ASYMPTOTIC EXPANSIONS OF CONFLUENT HYPERGEOMETRIC FUNCTIONS $U(a, c, z)$ FOR LARGE $|c|$.

7.1 The integro-differential equation.

The confluent hypergeometric or Kummer equation is

$$z y'' + (c - z) y' - a y = 0, \tag{1}$$

(cf. Slater 1960, §1.1.1; in the interests of consistent notation, references to long-standing work are given throughout to this monograph rather than to original papers). In the present chapter we study by methods similar to those employed in chapters 4, 5 and 6, asymptotic expansions of a pair of solutions of (1) for large $|c|$. Kummer's equation is related to the Whittaker equation (cf. (6.1.1))

$$\frac{d^2 W}{dz^2} = \left(\frac{1}{4} - \frac{k}{z} + \frac{m^2 - \frac{1}{4}}{z^2}\right) W, \tag{2}$$

by the transformations

$$y = z^{-\frac{1}{2}c} e^{\frac{1}{2}z} W, \tag{3}$$

$$a = \tfrac{1}{2} + m - k, \tag{4}$$

$$c = 1 + 2m. \tag{5}$$

The equivalent problem in Whittaker's notation is therefore k and m both large. Kazarinoff (1957), adopting Langer's methods, bases his calculations on (2) but, to facilitate application of our methods, Kummer's notation will be adhered to here.

In the limit $|z| \to \infty$, the behaviour of the solutions of (1) is determined by

$$z y'' - z y = 0, \tag{6}$$

two independent solutions of which are A and Be^{z}, where A and B are arbitrary constants. Let the solution of (1) that is regular at infinity be represented by $F_1(a, c, z)$, and the exponentially large solution by

$$\tilde{\mathcal{F}}_2(a,c,z) = e^{z} F_2(a,c,z) \tag{7}$$

Substitution into (1) then shows that F_2 satisfies the differential equation

$$z F_2'' + (z+c)F_2' + (c-a)F_2 = 0 . \tag{8}$$

Under the transformations

$$z \to -z,\ a \to 1-a,\ c \to 2-c,\ F_2 = z^{1-c}\, y, \tag{9}$$

(8) goes over into (1). It is therefore, in general, only necessary to treat F_1 in detail, and attention is confined to this function.

We introduce a parameter ρ and rewrite (1) in the form

$$(z-c+\rho)y' + ay = zy'' + \rho y' . \tag{10}$$

This parameter will later be chosen to simplify the relation between the expansions for F_1 and F_2. If the right-hand side of (10) is neglected, $y \sim (z-c+\rho)^{-a}$. Excluding the region where $z-c+\rho$ is small, for the present, we take this as the leading term in the expansion for y (or F_1). This is now seen to be justifiable, since in this approximation (and with $|c|$ large and $|a|$ not large) the term zy'' is indeed small compared with the individually large terms --- nearly cancelling --- on the left-hand side of (10). It is therefore expedient to write

$$F_1 = \frac{v}{(z-c+\rho)^a} . \tag{11}$$

Substitution into the complete equation, (10) then yields the integro-differential equation

$$v = \int^z \frac{1}{(z-c+\rho)} \left\{ zv'' + \frac{z(\rho-2a)-\rho(c-\rho)}{z-c+\rho} v' + \frac{az(a-\rho+1)+\rho a(c-\rho)}{(z-c+\rho)^2} v \right\} dz . \tag{12}$$

A suitable subsidiary independent variable, q say, reducing (12) to polynomial form is

$$q = \frac{\rho - c}{z - c + \rho} . \tag{13}$$

The parameter ρ will be chosen such that q remains unaltered under transformations (9). From the many possibilities we select $\rho = 2a$, as this choice leads to a considerably simplified integro-differential equation, and moreover furnishes a particularly suitable variable for the asymptotic expansions of the general exponential integral function to be considered in detail in § 7.6. If furthermore we write

$$b = c - 2a , \tag{14}$$

(12) contracts to

$$bv = q^2(1-q)\frac{dv}{dq} + (1-2a)q^2 v - (a-1)\int_0^q \left\{(a-2)q + a\right\} v \, dq, \tag{15}$$

where the lower limit of integration has been chosen so as to lead to easily identifiable solutions, and we shall in future write v' for dv/dq .

7.2 Green-type expansions.

As in e.g. §§4.2 and 5.3, direct application of the method of successive approximations to the integro-differential equation (7.1.15) yields a Green-type expansion, here in inverse powers of b . The zero-order approximation is taken to be $v_0 = 1$, and later contributions $v_i \ (i = 1, 2, 3, \ldots)$ are determined from the recurrence relation

$$bv_{i+1} = q^2(1-q)v_i' + (1-2a)q^2 v_i - (a-1)\int_0^q \{(a-2)q + a\} v_i \, dq. \quad (1)$$

The first few v_i are:-

$$v_1 = -\frac{aq}{2b}\{(a+1)q + 2(a-1)\},$$

$$v_2 = \frac{a(a+1)q^2}{24b^2}\{3(a+2)(a+3)q^2 + 4(a+2)(3a-5)q + 12(a-1)(a-2)\},$$

$$V_3 = -\frac{a(a+1)(a+2)}{48b^3} q^3 \Big\{ (a+3)(a+4)(a+5) q^3 + 2(a+3)(a+4)(3a-7) q^2$$

$$+ 4(a+3)(3a^2-13a+13) q + 8(a-1)(a-2)(a-3) \Big\},$$

$$V_4 = \frac{a(a+1)(a+2)(a+3)}{5760 b^4} q^4 \Big\{ 15(a+4)(a+5)(a+6)(a+7) q^4$$

$$+ 120(a+4)(a+5)(a+6)(a-3) q^3$$

$$+ 40(a+4)(a+5)(9a^2-51a+68) q^2$$

$$+ 96(a+4)(5a^3-40a^2+100a-77) q$$

$$+ 240(a-1)(a-2)(a-3)(a-4) \Big\}. \qquad (2)$$

Corresponding results for F_2 follow by (7.1.9).

It is at this stage convenient to identify the solutions. Equations (7.1.11) and (2) show that the behaviour of F_1 in the limit $|z| \to \infty$ $(q \to 0)$ is

$$F_1 = z^{-a}\left\{1 + O\left(|z|^{-1}\right)\right\}. \qquad (3)$$

Thus in Slater's notation (1960, § 4.1.2),

$$F_1 \,(a, c, z) = U(a, c, z) \qquad (4)^{+}$$

By (7.1.7) and (7.1.9), the second solution of (7.1.1) is then

$$\mathcal{F}_2(a, c, z) = e^{z} z^{1-c}\, U(1-a, 2-c, -z), \qquad (5)$$

denoted by y_8 on p.5 of Slater's monograph.

7.3 The Mellin transform of V.

The Green-type expansions break down in the 'transitional' region $z \sim c - 2a$. However, the Mellin transform method used in earlier chapters can be effect-

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+ In Erdélyi et al. (1953-5, ch. IV) this function is denoted by $\psi(a, c, z)$.

ively applied to provide expansions for F_1 (and $\mathcal{F}_2$) valid in this region.

Let $M(\mu)$, the Mellin transform of v , be defined by

$$v = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M(\mu)\left(\frac{2q^2}{b}\right)^{-\mu} d\mu . \qquad (1)$$

Substitution into (7.1.15) leads to the difference equation

$$(\mu - \tfrac{1}{2}a + \tfrac{1}{2})(\mu - \tfrac{1}{2}a + 1)M(\mu+1) - \mu M(\mu) =$$

$$\left(\frac{2}{b}\right)^{\frac{1}{2}}(\mu + \tfrac{1}{2}a)(\mu - \tfrac{1}{2}a + \tfrac{1}{2})M(\mu + \tfrac{1}{2}) , \qquad (2)$$

which has been written in a form suitable for solving by iteration. The zero-order approximation to $M(\mu)$ satisfies

$$(\mu - \tfrac{1}{2}a + \tfrac{1}{2})(\mu - \tfrac{1}{2}a + 1)M_0(\mu+1) - \mu M_0(\mu) = 0 , \qquad (3)$$

a solution of which is

$$M_0(\mu) = (\mu-1)!\,\left(-\mu+\tfrac{1}{2}a-1\right)!\,\left(-\mu+\tfrac{1}{2}a-\tfrac{1}{2}\right)! \tag{4}$$

If now a function $m(\mu)$ is defined by

$$M(\mu) = m(\mu)\, M_0(\mu), \tag{5}$$

successive contributions are given by the sum-equation

$$m_{i+1}(\mu) = -\left(\frac{2}{b}\right)^{\frac{1}{2}} \mathop{S}\left(\mu+\tfrac{1}{2}a\right) \frac{\left(\mu-\frac{1}{2}\right)!}{\mu!}\, m_i\left(\mu+\tfrac{1}{2}\right), \tag{6}$$

where the summation operator S is as defined in e.g. Nörlund (1923, ch.III). Agreement with the multiplying factor in the Green-type expansion for V is achieved by taking $m_0(\mu) = \left\{\left(\frac{1}{2}a-1\right)!\left(\frac{1}{2}a-\frac{1}{2}\right)!\right\}^{-1}$. Equations (6) and (5) then yield

$$M(\mu) = 2^{2\mu}\frac{(-2\mu+a-1)!}{(a-1)!}\left[(\mu-1)! - \frac{1}{3}\left(\frac{2}{b}\right)^{\frac{1}{2}}(2\mu+3a-2)\left(\mu-\tfrac{1}{2}\right)!\right.$$

$$+\frac{1}{18}\left(\frac{2}{b}\right)\left\{4\mu^2+3(4a-3)\mu+9a^2-15a+5\right\}\mu!$$

$$-\frac{1}{1620}\left(\frac{2}{b}\right)^{\frac{3}{2}}\left\{80\mu^3+60(6a-5)\mu^2\right.$$

$$\left.+2(270a^2-495a+179)\mu+270a^3-810a^2+675a-138\right\}\left(\mu+\tfrac{1}{2}\right)! \quad (7)$$

where the duplication formula for factorial functions has been used.

7.4 Transitional expansions.

Combination of (7.3.1) and (7.3.7) leads to a transitional expansion for V if the contour of integration, chosen such that it separates the poles of $(-2\mu+a-1)!$ from those of the factorials in square brackets in (7.3.7), $(\mu-1)!$, $(\mu-\frac{1}{2})!$, etc., is closed in a clockwise direction. The contributing poles are those of $(-2\mu+a-1)!$ at $\mu=\frac{1}{2}(a+n)$, where $n=0,1,2,\ldots$. Evaluation of the residues at these poles+ yields:-

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+ The Green-type expansion (7.2.2) is regained on closing the contour of integration specified above in an anti-clockwise direction, and evaluating the residues at the enclosed poles, i.e. at $\mu=-n$, $\mu=-n-\frac{1}{2}$, etc., where $n=0,1,2,\ldots$.

$$V = \frac{2^{a-1}}{(a-1)!} \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} \left(\frac{2\alpha^2}{b}\right)^{-\frac{1}{2}(a+n)} \left[\left\{ \tfrac{1}{2}(n+a-2) \right\}! \right.$$

$$- \frac{1}{3}\left(\frac{2}{b}\right)^{\frac{1}{2}} (n+4a-2)\left\{\tfrac{1}{2}(n+a-1)\right\}!$$

$$+ \frac{1}{36}\left(\frac{2}{b}\right)\left\{2n^2+(16a-9)n+32a^2-39a+10\right\} \left\{\tfrac{1}{2}(n+a)\right\}!$$

$$- \frac{1}{1620}\left(\frac{2}{b}\right)^{\frac{3}{2}}\left\{10n^3+(120a-75)n^2+(480a^2-645a+179)n\right.$$

$$\left.\left. + 640a^3-1380a^2+854a-138\right\}\left\{\tfrac{1}{2}(n+a+1)\right\}! + \cdots \right]. \quad (1)$$

7.41 Transition point.

The expansion for $\mathcal{U}(a,c,z)$ precisely at the transition point $z = c-2a$ is, by (7.1.11), (7.2.4) and (7.4.1),

$$\mathcal{U}(a,c,c-2a) = \frac{1}{2}\frac{e^{i\pi a}}{(a-1)!}\left(\frac{2}{b}\right)^{\frac{1}{2}a}\left[\left\{\tfrac{1}{2}(a-2)\right\}!\right.$$

$$- \frac{4}{3}\left(\frac{2}{b}\right)^{\frac{1}{2}}\left(a-\tfrac{1}{2}\right)\left\{\tfrac{1}{2}(a-1)\right\}! + \frac{1}{36}\left(\frac{2}{b}\right)\left(32a^2-39a+10\right)\left(\tfrac{1}{2}a\right)!$$

$$- \frac{1}{810}\left(\frac{2}{b}\right)^{\frac{3}{2}}\left\{320a^3 - 690a^2 + 427a - 69\right\}\left\{\tfrac{1}{2}(a+1)\right\}! + \cdots \Bigg]. \quad (1)$$

Corresponding results for $\mathcal{F}_2(a, c, z)$ follow directly from (7.2.5), (7.4.1) and (1).

7.5 Uniform expansions.

In the present section, the inverse Mellin transforms of the terms in (7.3.7) are expressed as a series involving a subsidiary function and its first derivative, thus yielding an expansion for V uniformly valid with respect to z.

Following §§ 4.5, 5.6 and 6.5, we select the subsidiary function from solutions of (7.1.15) for very large values of q. As indicated by (7.2.2), such a function necessarily involves a as a parameter. (As shown in the appendix, derivation of the dominant term of $\mathcal{U}(a, c, z)$ near the transition point leads to the same conclusion.) Retaining only the terms in q^2 on the right-hand side of (7.1.15), and introducing a new independent variable $x = 2q^2/b$, we obtain the following subsidiary function+:

$$k_a(x) = x^{-\frac{1}{2}a}\, U\left(\tfrac{1}{2}a, \tfrac{1}{2}, \tfrac{1}{x}\right), \tag{1}$$

essentially a parabolic cylinder function of order $\frac{1}{2}a$ (Slater, § 5.7). This choice has the merit that the inverse Mellin transform of the leading term in (7.3.7) is just $k_a(x)$, as is clear from the discussion leading up to (7.3.4), or from Slater, § 3.3.1. Calculation of the inverse Mellin transforms of the second and fourth quantities in (7.3.7) is simplified if these are brought into the form $\mathcal{A}\, P(\mu)$, where $P(\mu)$ is a polynomial in μ, and $\mathcal{A}$ an integral operator such that

$$\mathcal{A}\, P(\mu) = \frac{1}{2\pi i\left(\frac{1}{2}a-1\right)!\left(\frac{1}{2}a-\frac{1}{2}\right)!} \times \int_{c-i\infty}^{c+i\infty} \left(\mu-\tfrac{1}{2}\right)!\left(-\mu+\tfrac{1}{2}a-1\right)!\left(-\mu+\tfrac{1}{2}a-\tfrac{3}{2}\right)!\, P(\mu)\, x^{-\mu}\, d\mu. \tag{2}$$

+ Alternatives are obtained by retaining or adding terms of lower degree in q on the right-hand side of (1.11). With $\nu = -(2b)^{1/2} q^{-1}$, examples are the Lommel function $S_{-a+1/2,\,1/2}(\nu)$ of Watson (1944, § 10.71), and the combination of general sine and cosine integrals of order a denoted by $Q_a(\nu)$ in Dingle (1955b). None of these are as extensively tabulated as $U(a/2, 1/2, 1/x)$ however.

In this notation it follows by a well-known property of Mellin transforms that

$$\left.\begin{aligned} x^{\frac{1}{2}}\, k_a(x) &= \mathfrak{A}\ 1, \\ x^{\frac{3}{2}}\, k_a'(x) &= -\,\mathfrak{A}\ (\mu+\tfrac{1}{2}), \\ x^{\frac{5}{2}}\, k_a''(x) &= \mathfrak{A}\ (\mu+\tfrac{1}{2})(\mu+\tfrac{3}{2}), \text{ etc.} \end{aligned}\right\} \quad (3)$$

It is now a straight-forward algebraic routine to express the Mellin inverses of the terms in question as a series involving k_a and its derivatives. The third term in (7.3.7) may be handled similarly: a suitable integral operator to choose, $\mathfrak{B}$ say, is defined by

$$\mathfrak{B}\, P(\mu) = \frac{1}{2\pi i\, (\frac{1}{2}a-1)!\, (\frac{1}{2}a-\frac{1}{2})!}$$

$$\times \int_{c-i\infty}^{c+i\infty} (\mu-1)!\, (-\mu+\tfrac{1}{2}a-1)!\, (-\mu+\tfrac{1}{2}a-\tfrac{1}{2})!\, P(\mu)\, x^{-\mu}\, d\mu. \quad (4)$$

Second and later derivatives of $k_a(x)$ are related to $k_a(x)$ and $k_a'(x)$ by the differential equation

$$k_a''(x) = -\frac{1}{4x^2}\left[\left\{(2a+3)x+1\right\}k_a'(x) + a(a+1)\,k_a(x)\right]. \tag{5}$$

Using the above aids and (5), we obtain after a lengthy calculation that

$$v = k_a(x)\left\{1 - \frac{2a}{3b}q\sum_{n=0}^{\infty}\left(\frac{b}{q^3}\right)^n \xi_n(a,x)\right\}$$

$$-\frac{4q}{3b}\,k_a'(x)\sum_{n=0}^{\infty}\left(\frac{b}{q^3}\right)^n \eta_n(a,x), \tag{6}$$

where the first few ξ_n and η_n are

$$\xi_0(a,x) = 2a-1,$$

$$\xi_1(a,x) = \frac{a+1}{48}\left\{(16a-7)x+4\right\},$$

$$\xi_2(a,x) = \frac{a+1}{4320}\left\{(320a^3-690a^2+427a-69)x^3\right.$$

$$+ (600\,a^2 - 210a - 21)x^2 + (280a + 50)x + 40 \Big\} ; \qquad (7a)$$

$$\eta_0(a,x) = (2a-1)x + 1 ,$$

$$\eta_1(a,x) = \frac{1}{48}\Big\{(48a^2 - 30a + 3)x^2 + (40a - 2)x + 8\Big\},$$

$$\eta_2(a,x) = \frac{1}{4320}\Big\{(320\,a^4 - 370\,a^3 - 263\,a^2 + 358a - 69)x^4$$

$$+ (1720\,a^3 - 555\,a^2 - 313\,a + 69)x^3 + (1740\,a^2 + 420a - 12)x^2$$

$$+ (640\,a + 220)x + 80\Big\}. \qquad (7b)$$

Corresponding results for $U(1-a, 2-c, -z)$ follow on transforming (6) according to $b \to -b$, $x \to -x$ and $a \to 1-a$.

7.51 <u>Tabulation of k_a and k_a'.</u>

For purposes of computation it is best to express $k_a(x)$ in terms of the more comprehensively tabulated function ${}_1F_1$. By Slater, § 1.3,

$$k_a(x) = \frac{(-\frac{1}{2})!}{(\frac{1}{2}a - \frac{1}{2})!}\, x^{-\frac{1}{2}a}\, {}_1F_1\left(\tfrac{1}{2}a, \tfrac{1}{2}, \tfrac{1}{x}\right)$$

$$+ \frac{(-\frac{3}{2})!}{(\frac{1}{2}a - 1)!}\, x^{-\frac{1}{2}a + \frac{1}{2}}\, {}_1F_1\left(\tfrac{1}{2}a + \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{1}{x}\right). \qquad (1)$$

Differentiation of (1) with respect to x, together with relations given in Slater, § 2.1, yield

$$K_a'(x) = \frac{(-\frac{1}{2})!}{(\frac{1}{2}a-\frac{1}{2})!}\, x^{-\frac{1}{2}a-2}\Big\{(\tfrac{1}{2}ax+1)\, {}_1F_1(\tfrac{a}{2}, \tfrac{1}{2}, \tfrac{1}{x})$$

$$+ (a-1)\; {}_1F_1(\tfrac{a}{2}, \tfrac{3}{2}, \tfrac{1}{x})$$

$$+ \frac{(-\frac{1}{2})!}{(\frac{1}{2}a-1)!}\, x^{-\frac{1}{2}a-\frac{1}{2}}\Big\{ {}_1F_1(\tfrac{1}{2}a+\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{x})$$

$$+ (a-2)\; {}_1F_1(\tfrac{1}{2}a+\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{1}{x})\Big\}. \qquad (2)$$

Tabulated values of the function ${}_1F_1(\alpha, \gamma, z)$ for $\gamma = \frac{1}{2}, \frac{3}{2}$ required in (8) and (9) may be extracted from the following:

British Association (1926) to 6-7 figures for $\gamma = \pm\frac{1}{2}, \pm\frac{3}{2}$, $\alpha = -4(1/2)+4$, $z = 0(0.1)1(0.2)3(0.5)8$.

British Association (1927) to 6 figures for $\gamma = \pm\frac{1}{2}, \pm\frac{3}{2}$, $\alpha = -4(\frac{1}{2})+4$, $z = 0(0.02)0.08, 0.15(0.1)0.95, 1.1(0.2)1.9$.

MacDonald (1949) to 6 figures for $\gamma = \frac{1}{2}(\frac{1}{2})2$, $\alpha = 0.001, 0.01, 0.05, 0.1(0.1)1, 0.25, 0.75$, $z = \frac{1}{2}(\frac{1}{2})8$.

National Bureau of Standards (1949) to 7 figures for $\gamma = \frac{1}{2}$, $\alpha = 1\frac{1}{2}(1)100\frac{1}{2}$, $z = 0(0.01)1$.

Rushton and Lang (1954) to 7 figures for $\gamma = \frac{1}{2}(\frac{1}{2})2\frac{1}{2}(1)4\frac{1}{2}$ and a range of integer and half-integer values of α up to 40 or 50, $z = 0.02(0.02)0.1(0.1)1(1)10(10)50, 100, 200$.

Slater (1960) to 8-9 figures for $\gamma = 0.1(0.1)1.0$, $\alpha = -1.0(0.1)1.0$, $z = 0.1(0.1)10.0$.

7.6 The general exponential integral.

For $Re(z) > 0$ the general exponential integral $Ei_p(z)$ is defined by (Placzek 1946, Busbridge 1950)

$$Ei_p(z) = \int_1^\infty e^{-zu} u^{-p} \, du. \tag{1}$$

The relation between $Ei_p(z)$ and $U(a, c, z)$ is (Slater, § 5.6)

$$Ei_p(z) = e^{-z} U(1, 2-p, z). \tag{2}$$

The Green-type expansion for $Ei_p(z)$ may be derived from (2) and (7.2.2), or directly from (7.2.1). With $a = 1$ and

$$q \to \alpha = \frac{p}{z+p}\,; \tag{3}$$

either approach yields

$$Ei_p(z) = \frac{e^{-z}}{z+p}\left\{1 + \frac{\alpha^2}{p} + \frac{\alpha^3(3\alpha-2)}{p^2} + \frac{\alpha^4(15\alpha^2-20\alpha+6)}{p^3} + \frac{\alpha^5(105\alpha^3-210\alpha^2+130\alpha-24)}{p^4}\cdots\right\}. \tag{4}^{+}$$

The transitional expansion valid when $z \sim -p$ is, by (7.4.1) and (2),

$$Ei_p(z) = e^{-z}\left(-\frac{1}{2p}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-2)^n}{n!}\left(-\frac{2\alpha^2}{p}\right)^{-\frac{1}{2}n}\Bigg[\left(\tfrac{1}{2}n-\tfrac{1}{2}\right)! + \tfrac{2}{3}\left(-\tfrac{2}{p}\right)^{\frac{1}{2}}\left(\tfrac{1}{2}n+1\right)! + \tfrac{1}{9}\left(-\tfrac{2}{p}\right)\left(n+\tfrac{1}{2}\right)\left(\tfrac{1}{2}n+\tfrac{3}{2}\right)! + \tfrac{1}{81}\left(-\tfrac{2}{p}\right)^{\frac{3}{2}}\left(n^2+\tfrac{1}{2}n-\tfrac{3}{5}\right)\left(\tfrac{1}{2}n+2\right)! + \cdots\Bigg]. \tag{5}$$

+ Expansions essentially equivalent to this result have been given by Blanch (1946) and Dingle (1955a, eqn.11).

Precisely at the transition point we have for $Re(\rho) < 0$

$$Ei_{-\rho}(\rho) = e^{-\rho}\left(\frac{\pi}{2\rho}\right)^{\frac{1}{2}}\left\{1 + \frac{2}{3}\left(\frac{2}{\pi\rho}\right)^{\frac{1}{2}} + \frac{1}{12\rho} + \frac{2\pi}{135}\left(\frac{2}{\pi\rho}\right)^{\frac{3}{2}}\right\}. \qquad (6)$$

Combination of (7.5.6), (7.5.7a), (7.5.7b) and (2) yields the uniform expansion

$$Ei_{\rho}(z) = \frac{e^{-z}}{z+\rho}\left[k_1(x)\left\{1 + \frac{2\alpha}{3\rho}\sum_{n=0}^{\infty}\left(\frac{\rho}{\alpha^3}\right)^n \theta_n(x)\right\}\right.$$

$$\left. + \frac{4\alpha}{3\rho}\, k_1'(x) \sum_{n=0}^{\infty}\left(\frac{\rho}{\alpha^3}\right)^n \phi_n(x)\right], \qquad (7)$$

where the first few terms are

$$\theta_0(x) = 1$$

$$\theta_1(x) = -\frac{1}{24}(9x+4),$$

$$\theta_2(x) = \frac{-1}{2160}\left(12x^3 - 369x^2 - 330x - 40\right), \qquad (8a)$$

$$\phi_0(x) = x+1,$$

$$\phi_1(x) = -\frac{1}{48}\left(21\ x^2 + 38x + 8\right),$$

$$\phi_2(x) = \frac{-1}{4320}\left(24x^4 - 921x^3 - 2148x^2 - 860x - 80\right). \quad (8b)$$

The subsidiary function $k_1(x)$ is expressible in terms of extensively tabulated functions, for by e.g. Erdélyi et al. (1953-5, § 6.9.2)

$$k_1(x) = 2x^{-\frac{1}{2}} e^{\frac{1}{x}} \operatorname{Erfc}(x^{-\frac{1}{2}}). \quad (9)^+$$

Moreover, by (9) and relations given in Erdélyi §§ 9.9 and 9.2,

$$k_1'(x) = -\left(\frac{1}{2x} + \frac{1}{x^2}\right) k_1(x) + \frac{1}{x^2}. \quad (10)$$

\+ Numerous tables exist for the function $\operatorname{Erfc}(z)$ and closely related quantities; perhaps the most convenient for the present purpose are those of Burgess (1898), Rosser (1947, 1948) and Karpov (complex plane, 1954).

Thus with the aid of (10), (7) may be re-written in the following simpler form:

$$Ei_p(z) = \frac{e^{-z}}{z+p}\left\{ k_1(x) \sum_{n=0}^{\infty}\left(-\frac{p}{\alpha^3}\right)^n \psi_n(x) + \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{p}{\alpha^3}\right)^{n+1}\phi_n(x)\right\}, \qquad (11)$$

where for the first few terms

$$\psi_0(x) = 1,$$

$$\psi_1(x) = \frac{1}{6}(3x+2),$$

$$\psi_2(x) = \frac{1}{288}(3x^3+72x^2+84x+16),$$

$$\psi_3(x) = \frac{1}{25920}(135x^4+3330x^3+5076x^2+1800x+160). \qquad (12)$$

7.7 Appendix. Behaviour of $\mathcal{U}(a,c,z)$ near a transition point, starting from its integral representation.

The integral representation of $\mathcal{U}(a,c,z)$ for $Re(a) > 0$ and $Re(z) > 0$ is (cf. Slater 1960, (3.1.19))

$$\mathcal{U}(a,c,z) = \frac{1}{(a-1)!} \int_0^\infty e^{-zt}\, t^{a-1} (1+t)^{c-a-1}\, dt. \qquad (1)$$

Considering $|c|$ large and grouping the fast-varying terms together in the exponent, we rewrite (1) as

$$\mathcal{U}(a,c,z) = \frac{1}{(a-1)!} \times \int_0^\infty e^{-zt + (c-\mu)\ln(1+t)}\, t^{a-1} (1-t)^{\mu-a-1}\, dt, \qquad (2)$$

where μ is an as yet arbitrary parameter. The stationary point of the exponent at

$$t = \frac{c-\mu}{z} - 1, \qquad (3)$$

lies outside the integration interval when $\mathrm{Re}\left(\frac{c-\mu}{z}-1\right) < 0$, and inside when $\mathrm{Re}\left(\frac{c-\mu}{z}-1\right) < 0$, while for the transition point value $z = c-\mu$, it lies exactly at $t = 0$. We are interested here in the behaviour of $U(a,c,z)$ near $z = c-\mu$ and therefore the main contribution to the value of the integral will come from small values of t. Expanding the terms in (2), taking $\mu = 2a$ and $b = c-\mu$, we obtain

$$U(a,c,z) = \frac{1}{(a-1)!}\int_0^\infty e^{-t(z-b)-\frac{1}{2}bt^2+b\left\{\ln(1+t)-t+\frac{1}{2}t^2\right\}}$$

$$\times\, t^{a-1}\left\{1+(a-1)t+\cdots\right\}\,dt. \qquad (4)$$

In the region of interest, the quadratic term in t in the exponent of (4) is comparable in magnitude to the term linear in t, and must thus be taken into account, even to the zero-order approximation, while the third term is $O(t^3)$ and may in this approximation be neglected. Near $z = c - 2a$, therefore,

$$\mathcal{U}(a,c,z) \simeq \frac{1}{(a-1)!} \int_0^\infty e^{-t(z-b)-\frac{1}{2}bt^2}\, t^{a-1}\, dt. \qquad (5)$$

Moreover, by Miller (1955, (4.13)), the function $\mathcal{U}(a-\frac{1}{2}, y)$ has the integral representation (for $Re(a) > 0$)

$$\mathcal{U}(a-\tfrac{1}{2}, y) = \frac{1}{(a-1)!}\, e^{-\frac{1}{4}y^2} \int_0^\infty e^{-ys-\frac{1}{2}s^2}\, s^{a-1}\, ds, \qquad (6)$$

which with $y = (z-b) / b^{1/2}$ and $s = t\, b^{1/2}$ becomes

$$\mathcal{U}(a-\tfrac{1}{2}, z-b/\sqrt{b}) = \frac{b^{\frac{1}{2}a}}{(a-1)!}\, e^{-(z-b)^2/4b}$$

$$\times \int_0^\infty e^{-(z-b)t-\frac{1}{2}bt^2}\, t^{a-1}\, dt. \qquad (7)$$

Hence with $x = 2b/(z-b)^2$ as in § 7.5, and the relation (cf. Miller 1955, p.73)

$$U\left(\tfrac{1}{2}a, \tfrac{1}{2}, \tfrac{1}{x}\right) = 2^{\frac{1}{2}a} e^{\frac{1}{2x}} U\left(a-\tfrac{1}{2}, \sqrt{\tfrac{2}{x}}\right), \tag{8}$$

(5) shows that for $z \sim c - 2a$

$$U(a,c,z) \sim \frac{1}{(z-b)^a} x^{-\frac{1}{2}a} U\left(\tfrac{1}{2}a, \tfrac{1}{2}, \tfrac{1}{x}\right), \tag{9}$$

thus substantiating a remark made in connection with (7.5.1).

8. ASYMPTOTIC EXPANSIONS OF THE MATHIEU FUNCTIONS $ce(z, h)$ AND $se(z, h)$ FOR LARGE h.

8.1 Genesis of Mathieu's differential equation.

One way of obtaining Mathieu's differential equation is by separating the wave equation

$$\nabla^2 \psi + k^2 \psi = 0 \tag{1}$$

in elliptic cylinder co-ordinates. The relations between the elliptic cylinder co-ordinates ξ, η and z and the Cartesian co-ordinates x, y and z are

$$\left.\begin{aligned} x &= c \cosh \xi \cos \eta, \\ y &= c \sinh \xi \sin \eta, \\ z &= z \end{aligned}\right\} \tag{2}$$

Here, $2c$ is the interfocal distance, and ξ and η are restricted to the ranges

$$0 \leq \xi \leq \infty , \; -\pi \leq \eta \leq \pi . \tag{3}$$

The wave equation (1) can be separated in elliptic cylinder co-ordinates by assuming solutions of the form

$$\psi = A(\xi)\, B(\eta)\, Z(z) . \tag{4}$$

Substitution of (4) into (1) yields the following equations:

$$\frac{d^2 A}{d \xi^2} - (\lambda - 2h^2 \cosh 2\xi) A = 0 , \tag{5a}$$

$$\frac{d^2 B}{d \eta^2} + (\lambda - 2h^2 \cos 2\eta) B = 0 , \tag{5b}$$

$$\frac{d^2 Z}{d z^2} = 0 , \tag{5c}$$

where λ is the separation constant and we have put $2h^2 = k^2 c^2/2$. The equation for B is known as Mathieu's equation. Since (5a) follows from (5b) by the transformation $\eta = \pm i\xi$, the equation for A is nowadays usually referred to as the modified Mathieu equation (cf. Jeffreys 1924, N.B.S. 1964, ch.20) in analogy with the relation between the Bessel and the modified Bessel equations. The designation Mathieu functions is generally reserved for those solutions of (5b) which have periodicity π or 2π. As we shall see in subsequent sections, such functions satisfy (5b) only for certain values of λ, termed eigenvalues, which depend on h and on the order of the functions.

8.2 Brief survey of earlier work, and applications of Mathieu's equation to problems in physics and engineering.

Mathieu's equation and its solutions were first studied by Mathieu (1868) in connexion with the determination of the natural frequencies for a stretched membrane of elliptical shape. As the eccentricity of the elliptic boundary tends to zero, the problem becomes essentially that of solving Bessel's equation. It is thus to be expected that there exists a close relationship between Mathieu and Bessel functions. In fact, Heine (1878) showed that one set of periodic solutions of (8.1.5b) could be expanded in a series of Bessel functions. Since those early days, Mathieu's equation has found application in a diversity of mathematical and physical problems.

From the mathematical point of view, the Mathieu equation is of considerable importance as the simplest representative of a host of differential equations with one or more periodic coefficients. These include, for instance, prolate and oblate spheroidal wave equations and Lamé's equation. To quote Professor Erdélyi in his foreword to the National Bureau of Standards tables (1951) : " Among all the special functions arising out of the separation of the partial differential equations of mathematical physics in various systems of coordinates, Mathieu functions are the first to lead definitely outside the circle of hypergeometric and allied functions; hence the difficulties encountered in their theory and numerical computation alike, and hence the necessity of using new methods."

In physical applications, Mathieu's differential equation arises essentially in one of two ways:

i) separation of the wave equation for a three-dimensional problem with elliptic cylindrical symmetry , or for a two-dimensional problem with elliptical symmetry; and

ii) study of one-dimensional problems involving periodically varying parameter(s). The following may be cited as examples of the first category: diffraction of electromagnetic waves by an elliptical cylinder (Sieger 1908), electromagnetic waves in metal cylinders of elliptic cross-section (Chu 1938), elliptic antennae (Schelkunoff 1952), free oscillations of water in elliptical lakes (Jeffreys 1924), stability of columns and strings under periodically varying forces (Lubkin and Stoker 1943), aerofoil in a windtunnel of elliptic section (Rosenhead 1933), motion of elliptic cylinders through viscous liquid (Harrison 1924), vibration of elliptic cylinder in viscous fluid (Ray 1936), steady flow of viscous fluid past an elliptic cylinder (Tomotika and Aoi 1953), skin effect in cylindrical conductors (Strutt 1928), wave mechanical theory of the H_2^+ ion (Teller 1930, Hylleraas 1931, Johnson 1941), thermodynamic functions for molecules having restricted internal rotations (Pitzer 1937, Li and Pitzer 1956), restricted rotations of molecules and tunnelling through periodic barriers (Das 1957, Stejskal and Gutowski 1958). Some problems belonging to the second class are: stability of periodic motion (Thomson 1892a,b), maintenance of vibrations (Raman 1912), frequency modulation (Carson 1922, Barrow 1934, Erdélyi 1934, Barrow, Smith and Baumann 1936), stability of non-linear oscillators (McLachlan 1951), the physical pendulum in wave mechanics (Condon 1928).

Several treatises on the theory and application of Mathieu functions have been published. The monograph by Strutt (1932) is useful for its extensive bibliography of papers published prior to 1932. In his book on the theory and application of Mathieu functions, McLachlan (1947) has sacrificed some mathematical rigour to produce a reference work which covers in useful detail the applied mathematics, physics and engineering aspects of Mathieu functions. An extensive bibliography is given in addition to many new results. The book by Meixner and Schäfke (1954): 'Mathieusche Funktionen und Sphäroidfunktionen', deals with the subject from a more analytical standpoint. There are also long chapters on Mathieu functions in volume III of the Bateman manuscript 'Higher transcendental functions' prepared by Erdélyi and others (1955). In addition, Mathieu functions are studied extensively in such texts as: 'A course in modern analysis' by Whittaker and Watson (1927) and 'Methods of theoretical physics' by Morse and Feshbach (1953). Some early tables on Mathieu functions, and their characteristic numbers, were published by Goldstein (1927), but the most extensive tables available for a long time were those of Ince (1932), which also include values of the zeros and turning points. More recent tables giving Fourier coefficients of Mathieu functions and their characteristic numbers are those of Stratton, Morse, Chu and Hutner (1941), and the National Bureau of Standards (1951; these tables also give joining factors relating to the various solutions of Mathieu's equation). In the Introduction to the latter work, Dr.G.Blanch gives an excellent account of the more practical aspects of the theory of Mathieu functions.

8.3 The integro-differential equation.

Let the Mathieu equation be written

$$\frac{d^2y}{dz^2} = \left(4h^2\cos^2 z - \lambda - 2h^2\right)y, \quad (-\pi \leq z \leq \pi), \tag{1}$$

where h^2 is large and positive and λ represents the characteristic numbers. The form of the asymptotic expansion for λ when h^2 is large is well-known (cf. e.g. Ince 1926, 1927; Goldstein 1927; Dingle and Müller 1962), and for later convenience we write

$$\lambda = -2h^2 + 2hm - \tfrac{1}{8}(m^2+1) + \sum_{j=0}^{\infty} \frac{a_j}{(8h)^j}, \tag{2}$$

where m is approximately an odd integer, and a_j are coefficients depending only on m. (We will show that $a_0 = 0$, so that the third term in (2) does in fact constitute the term of $O(h^0)$ in the expansion for λ.)

By (2), when h^2 is large the leading terms on the right-hand side of (1) are $(4h^2\cos^2 z - 2hm)y$, in which the first is predominant except where $|z| \stackrel{\Omega}{=} \frac{1}{2}\pi$. Excluding this region for the present, we obtain as zero-order approximations to a pair of independent solutions (cf. § 3.2)

$$y_{\pm}^{0} \propto \left(4h^2\cos^2 z\right)^{-1/4} \exp\left[\pm\int^{z}\left(4h^2\cos^2 z\right)^{\frac{1}{2}}\left\{1-\frac{m}{4h\cos^2 z}\right\}dz\right]. \quad (3)$$

Noting that (1) is even in z, we are led to write a pair of solutions as

$$y_1(z) = (\cos z)^{-\frac{1}{2}} e^{2h\sin z}\left(\frac{1+\sin z}{1-\sin z}\right)^{-\frac{1}{4}m} V(-z), \qquad (4)$$

$$y_2(z) = (\cos z)^{-\frac{1}{2}} e^{-2h\sin z}\left(\frac{1+\sin z}{1-\sin z}\right)^{\frac{1}{4}m} V(z), \qquad (5)$$

where $V = 1 + O\left(h^{-1}\right)$. Henceforth it is therefore only necessary to consider one of these solutions, y_2 say, in detail.

Substitution of (5) into (1) leads to the following integro-differential equation for $V(z)$:

$$4h\,v = \frac{1}{\cos z}\,\frac{dv}{dz} + \frac{m}{\cos^2 z}\,v$$

$$+\frac{1}{16}\int^{z}\left\{\frac{(m-3)(m-1)\cos z}{(1-\sin z)^2} + 16\sum_{j=0}^{\infty}\frac{a_j}{(8h)^j}\,\frac{1}{\cos z} + \frac{(m+3)(m+1)\cos z}{(1+\sin z)^2}\right\}v\,dz, \tag{6}$$

which can be reduced to polynomial form by the introduction of the new independent variable

$$q = \frac{1+\sin z}{1-\sin z}\,. \tag{7}$$

With λ expressed as in (2), the integro-differential equation for $v(q)$ becomes

$$8h\,v = (q+1)^2\,v' + \tfrac{1}{2}m\,\frac{(q+1)^2}{q}\,v$$

$$+\frac{1}{16}\int^{q}\left\{(m-3)(m-1) + 16\sum_{j=0}^{\infty}\frac{a_j}{(8h)^j}\,\frac{1}{q} + (m+3)(m+1)\frac{1}{q^2}\right\}v\,dq, \tag{8}$$

where the prime signifies differentiation with respect to q .

An expansion for v in inverse powers of h follows on solving (8) by the method of successive approximations. If v_i is the ith-order term in this expansion, the next higher order term v_{i+1} satisfies the recurrence relation:

$$8hv_{i+1} = (q+1)^2 v_i' + \frac{m(q+1)^2}{2q} v_i$$

$$+ \frac{1}{16} \int^q \left\{ (m-3)(m-1) + (m+3)(m+1)\frac{1}{q^2} \right\} v_i \, dq$$

$$+ \sum_{j=0}^{i} \frac{a_j}{(8h)^j} \int^q \frac{1}{q} v_{i-j} \, dq \, . \qquad (9)$$

To obtain the simplest possible expansion for v, the lower limits of integration in (9) will be chosen such that no constants appear in $v_{i \neq 0}$. With $v_0 = 1$, (9) then yields

$$v_1 = \frac{1}{2^7 h} \left\{ (m+3)(m+1) q - (m-3)(m-1)\frac{1}{q} \right\}$$

$$+ \frac{1}{2^3 h} a_0 \ln q \, . \qquad (10)$$

Periodicity and uniqueness of the solutions prescribe that there must be no logarithmic terms in V (cf. Ince 1926, Goldstein 1927), so that $a_0 = 0$. As will become apparent, a general coefficient a_{i-1} in the expansions for the characteristic numbers is determined concurrently with the contribution V_i in the expansions for the functions themselves.

Substitution for V_1 into (9) yields

$$V_2 = \frac{1}{2^{15} h^2} \left\{ (m+1)(m+3)(m+5)(m+7)\, q^2 + 32(m+1)(m+2)(m+3)\, q - 32(m-1)(m-2)(m-3)\frac{1}{q} + (m-1)(m-3)(m-5)(m-7)\frac{1}{q^2} \right\}, \quad \text{(11a)}$$

where the condition on the coefficient of the logarithmic term is satisfied for

$$a_1 = -\frac{m(m^2+3)}{16}. \quad \text{(11b)}$$

Similarly,

$$V_3 = \frac{1}{3\cdot 2^{22} h^3}\Big\{(m+1)(m+3)(m+5)(m+7)(m+9)(m+11)\,q^3$$

$$+96(m+1)(m+3)^2(m+5)(m+7)\,q^2$$

$$+3(m+1)(m+3)(m^4-8m^3+726m^2+2792m+3273)q$$

$$-3(m-1)(m-3)(m^4+8m^3+726m^2-2792m+3273)\frac{1}{q}$$

$$+96(m-1)(m-3)^2(m-5)(m-7)\frac{1}{q^2}$$

$$-(m-1)(m-3)(m-5)(m-7)(m-9)(m-11)\frac{1}{q^3}\Big\}, \quad (12a)$$

and

$$a_2 = -\frac{1}{2^6}\left(5m^4+34m^2+9\right). \qquad (12b)$$

The transformation $z \to -z$ is by (7) equivalent to $q \to 1/q$, which in (8) corresponds to $(m, h) \longrightarrow (-m, -h)$. Accordingly, for the second solution,

$$v(-z) = v(q, -m, -h). \tag{13}$$

The successive contributions to v form a rapidly decreasing sequence provided

$$|h^{-1}| << |q| << |h|. \tag{14}$$

The expansions break down when q is either very large or very small, i.e. when $|z| \simeq \frac{1}{2}\pi$. In form, the above expansions correspond closely to those derived in § 2 of Dingle and Müller (1962) by a perturbation technique.

8.4 <u>The Mellin transform of v</u>.

To obtain expansions of less restricted validity than those of the previous section, (8.3.8) will now be solved by the Mellin transform technique.

The Mathieu functions for real z are all real, but for convenience we carry the calculations through for a general solution of (8.3.8), V say, of which v is the real part. Let $M(\mu)$, the Mellin transform of V, be defined by

$$V = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M(\mu)\left(-\frac{q}{8h}\right)^{-\mu} d\mu . \tag{1}$$

Substitution for V into (8.3.8) yields the difference equation

$$\left(\mu-\frac{m-3}{4}\right)\left(\mu-\frac{m-1}{4}\right)M(\mu+1)-\mu M(\mu) =$$

$$\frac{M(\mu)}{8h}\left\{2\mu^2-m\mu+\sum_{j=0}^{\infty}\frac{a_j}{(8h)^j}\right\}-\frac{M(\mu-1)}{(8h)^2}\left(\mu-\frac{m+3}{4}\right)\left(\mu-\frac{m+1}{4}\right), \tag{2}$$

which has been written in a form suitable for solving by successive approximations.

With $h \to \infty$, we obtain the zero-order approximation

$$M_0(\mu) = (\mu-1)!\left(-\mu+\frac{m-1}{4}\right)!\left(-\mu+\frac{m-3}{4}\right)! . \tag{3}$$

If now a function $m(\mu)$ is defined by

$$M(\mu) = M_0(\mu)\, m(\mu), \tag{4}$$

successive contributions to $m(\mu)$ are given by the sum-equation

$$m_{i+1}(\mu) = \frac{1}{8h} \mathop{S} (2\mu - m)\, m_i(\mu)$$

$$+ \sum_{j=0}^{i} \frac{a_j}{(8h)^{j+1}} \mathop{S} \frac{m_{i-j}(\mu)}{\mu}$$

$$- \frac{1}{(8h)^2} \mathop{S} \frac{\left(\mu - \frac{m+3}{4}\right)^2 \left(\mu - \frac{m+1}{4}\right)^2}{\mu(\mu-1)}\, m_{i-1}(\mu-1), \quad (5)$$

where the summation operator $\mathop{S}$ is as defined in e.g. Nörlund (1923, ch.III). The initial constant term $m_0(\mu)$ and the summation constants will be chosen so as to achieve agreement with the expansion for given in § 8.3. Therefore $m_0(\mu)$ is taken equal to $\left[\left(\frac{1}{4}m - \frac{1}{4}\right)!\left(\frac{1}{4}m - \frac{3}{4}\right)!\right]^{-1}$. Equation (5) then yields

$$m_1(\mu) = \frac{1}{\left(\frac{m-1}{4}\right)!\left(\frac{m-3}{4}\right)!\, 8h} \left\{ \mu(\mu - m - 1) + a_0 \psi(\mu - 1) \right\}, \quad (6)$$

where $\psi(\mu) = (d/d\mu) \ln \mu!$. Retention of $\psi(\mu-1)$ in $m_1(\mu)$ would lead to double and higher-order poles in $M(\mu)$, and consequently to logarithmic terms in expansions of the type derived in § 8.3. As we have already remarked, such terms violate the periodicity and uniqueness conditions for Mathieu functions. Accordingly $a_0 = 0$, as in § 8.3. Continuing the iteration, and again setting the coefficient of $\psi(\mu-1)$ equal to zero, we obtain

$$m_2(\mu) = \frac{1}{\left(\frac{m-1}{4}\right)!\left(\frac{m-3}{4}\right)!(8h)^2}\left[\tfrac{1}{2}\mu(\mu-1)\left\{\mu^2-(2m+3)\mu+(m+1)(m+2)\right\}\right.$$

$$\left. -\frac{\mu}{8}\left(3m^2+4m+3\right)+\frac{(m-3)^2(m-1)^2}{16^2}\,\frac{\mu}{\mu-1}\right], \quad (7)$$

with $a_1 = -m(m^2+3)/2^4$, as in (8.3.11b).

Calculation of the third-order term yields a_2 as in (8.3.12b). Thus by (4), to $O(h^{-3})$,

$$M(\mu) = \frac{\left(-\mu+\frac{m-1}{4}\right)!\left(-\mu+\frac{m-3}{4}\right)!}{\left(\frac{m-1}{4}\right)!\left(\frac{m-3}{4}\right)!}\left[\;(\mu-1)!\right.$$

$$+ \frac{1}{8h}\left(\mu - m - 1\right)\mu!$$

$$+ \frac{1}{2^9 h^2}\left\{4\mu^3 - 8(m+2)\mu^2 + 4(m^2+5m+5)\mu - (7m^2+16m+11)\right\}\mu!$$

$$+ \frac{1}{2^{14}h^2}(m-3)^2(m-1)^2\mu(\mu-2)!$$

$$+ \frac{1}{3\cdot 2^{17}h^3}\left\{128\mu^5 - 384(m+3)\mu^4 + 128(3m^2+21m+31)\mu^3 - 32(4m^3+69m^2+224m+213)\mu^2 + (3m^4+648m^3+4386m^2+8664m+6011)\mu - (3m^5-24m^4+994m^3+3384m^2+5243m+2688)\right\}\mu!$$

$$+ \frac{1}{2^{17}h^3}(m-3)^2(m-1)^2(m-2)\mu(\mu-2)! + O(h^{-4})\Bigg]. \qquad (8)$$

8.5 Expansions for $|z| \simeq \frac{1}{2}\pi$.

The separation constant c in the Mellin-Barnes integrals for V is chosen such that the line from

$c - i\infty$ to $c + i\infty$ separates the poles of $\left(-2\mu + \frac{m-1}{2}\right)!$ from the poles of the factorial functions in square brackets in (8.4.8). When the contour of integration is closed in an anti-clockwise sense, calculation of the residues at the enclosed poles leads to the expansions already derived in § 8.3.

When the contour is closed in a clockwise sense, calculation of the residues at the enclosed poles yields expansions which are valid in the region where the expansions of § 8.3 break down (i.e. for $|z| \simeq \frac{1}{2}\pi$). Since $v = \mathrm{Re}(V)$, (8.4.1) and (8.4.8) yield

$$v\left(z \simeq \tfrac{1}{2}\pi\right) = \frac{\sqrt{\pi}}{\left(\frac{m-1}{4}\right)!\left(\frac{m-3}{4}\right)!}\left(\frac{8h}{q}\right)^{\frac{1}{4}(m+1)}$$

$$\times \sum_{n=0}^{\infty} \frac{(-2)^n \cos\frac{\pi}{4}(2n+m+1)}{n!}\left(\tfrac{1}{2}n + \frac{m-3}{4}\right)!\left(\frac{8h}{q}\right)^{\frac{1}{2}n}$$

$$\times \left[1 + \frac{1}{2^7 h}\left\{4n^2 - 4(m+1)n - 3(m+1)^2\right\}\right.$$

$$+ \frac{1}{2^{15}h^2}\left\{16n^4 - 32(m+3)n^3 - 8(m^2-10m-19)n^2\right.$$

$$+ 8(3m^3 - m^2 - 11m - 15)n$$

$$+ (9m^4 - 36m^3 - 178m^2 - 244m - 111)$$

$$+ 2(m-3)^2(m-1)^2 \frac{(2n+m+1)}{(2n+m-3)} \Bigg\}$$

$$+ \frac{1}{3 \cdot 2^{22} h^3} \Bigg\{ 64n^6 - 192(m+5)n^5 + 16(3m^2 + 126m + 331)n^4$$

$$+ 32(7m^3 - 27m^2 - 259m - 441)n^3$$

$$- 4(3m^4 + 276m^3 - 654m^2 - 3180m - 4429)n^2$$

$$- 4(33m^5 - 315m^4 - 94m^3 + 498m^2 + 3949m + 2937)n$$

$$- 3(15m^6 - 206m^5 + 545m^4 + 4220m^3 + 10561m^2 + 11250m + 4143)$$

$$+ 3(m-3)^2(m-1)^2(m-2) \frac{(2n+m+1)}{(2n+m-3)} \Bigg\} + O(h^{-4}) \Bigg]. \quad (1)$$

The expansion for $y_2(z)$ precisely at $z = \frac{1}{2}\pi \; (q \to \infty)$ is by (8.3.5), (8.3.7) and (1),

$$y_2\left(z=\tfrac{1}{2}\pi\right) = -\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{(8h)^{\frac{1}{4}(m+1)} \sin\frac{\pi}{4}(m-1)}{e^{2h}\left(\frac{m-1}{4}\right)!}$$

$$\times \left\{ 1 - \frac{3(m+1)^2}{2^7 h} + \frac{1}{2^{15}h^2}\left(11m^4 - 44m^3 - 174m^2 - 236m - 117\right) \right.$$

$$- \frac{1}{2^{22}h^3}\left(15m^6 - 207m^5 + 551m^4 + 4210m^3\right.$$

$$\left. + 10561m^2 + 11261m + 4137\right)$$

$$\left. + O\left(h^{-4}\right) \right\}. \qquad (2)$$

It is clear from (8.4.1) that $V(q,-m,-h)$ is a real function, so that the expansion for $y_1(z)$ valid near $z = \frac{1}{2}\pi$ is, by (8.3.13) and (1),

$$y_1\left(z \simeq \tfrac{1}{2}\pi\right) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{e^{2h\sin z}\left(1+\frac{1}{q}\right)^{\frac{1}{2}}}{(8h)^{\frac{1}{4}(m-1)}\left(-\frac{m+1}{4}\right)!\left(-\frac{m+3}{4}\right)!}$$

$$\times \sum_{n=0}^{\infty} \frac{(-2)^n}{n!}\left(\tfrac{1}{2}n - \frac{m+3}{4}\right)!\left(\frac{8h}{q}\right)^{\frac{1}{2}n}\left[1 - \right.$$

$$- \frac{1}{2^7 h} \left\{ 4n^2 + 4(m-1)n - 3(m-1)^2 \right\}$$

$$+ \frac{1}{2^{15} h^2} \left\{ 16n^4 + 32(m-3)n^3 - 8(m^2+10m-19)n^2 \right.$$

$$- 8(3m^3 + m^2 - 11m + 15)\,n$$

$$+ (9m^4 + 36m^3 - 178m^2 + 244m - 111)$$

$$\left. + 2(m+3)^2(m+1)^2 \frac{(2n-m+1)}{(2n-m-3)} \right\}$$

$$- \frac{1}{3 \cdot 2^{22} h^3} \left\{ 64n^6 + 192(m-5)n^5 + 16(3m^2 - 126m + 331)n^4 \right.$$

$$- 32(7m^3 + 27m^2 - 259m + 441)\,n^3$$

$$- 4(3m^4 - 276m^3 - 654m^2 + 3180m - 4429)\,n^2$$

$$+ 4(33m^5 + 315m^4 - 94m^3 - 498m^2 + 3949m - 2937)\,n$$

$$- 3(15m^6 + 206m^5 + 545m^4 - 4220m^3 + 10561m^2 - 11250m + 4143)$$

$$\left. \left. - 3(m+3)^2(m+1)^2(m+2) \frac{(2n-m+1)}{(2n-m-3)} \right\} + O\left(h^{-4}\right) \right] . \quad (3)$$

In particular,

$$y_1\left(z=\tfrac{1}{2}\pi\right)=\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{e^{2h}}{(8h)^{\frac{1}{4}(m-1)}\left(-\frac{m+1}{4}\right)!}\left[1+\frac{3(m-1)^2}{2^7 h}\right.$$

$$+\frac{11m^4+44m^3-174m^2+236m-117}{2^{15}h^2}$$

$$+\frac{1}{2^{22}h^3}\left(15m^6+207m^5+551m^4-4210m^3\right.$$

$$\left.+10561m^2-11261m+4137\right)$$

$$\left.+O(h^{-4})\right]. \tag{4}$$

The case $z \stackrel{\Omega}{=} -\frac{1}{2}\pi$ need not be considered separately, since (8.3.4-5) imply that

$$y_2(-z)=y_1(z),\quad y_1(-z)=y_2(z). \tag{5}$$

These new expansions have comparable validity to those given by Meixner (1948) and Sips (1949) in terms of parabolic cylinder functions, and by Dingle and Müller (1962) in terms of Hermite functions+, but they

+ There is a notational inconsistency in § 5 of the latter reference, in that $C(z)$ is used to represent first a complex function and subsequently its real part. This could have been avoided if $C(z)$ had been defined as the real part of the function on the right of (46).

have the important advantage of involving only elementary functions (cf. § 8.71).

8.6 Uniform expansions.

The expansions given in §§ 8.3 and 8.5 between them cover practically the whole range of z . The uniform expansion, including these expansions as special cases, is obtained by expressing the Mellin-Barnes integrals of (8.4.1) in terms of a 'flexible' subsidiary function and its first derivative. As indicated in e.g. §§ 4.5 and 5.6, a suitable function, $k_m(x)$ say, is defined by the inverse Mellin transform of the first term in (8.4.8). Considering only real functions we therefore choose

$$k_m(x) = \mathcal{R}e \frac{1}{2\pi i \left(\frac{m-3}{4}\right)! \left(\frac{m-1}{4}\right)!} \times \int_{c-i\infty}^{c+i\infty} (\mu-1)! \left(-\mu+\frac{m-1}{4}\right)! \left(-\mu+\frac{m-3}{4}\right)! \, x^{-\mu} \, d\mu, \tag{1}$$

where $x = -q/8h$, and $\mathcal{R}e$ stands for 'real part'. This function is closely related to the subsidiary function selected in § 7.5 , in that it is essentially a Weber parabolic cylinder function.

Application of techniques given in e.g. § 4.5 enables the remaining Mellin-Barnes integrals of (8.4.1) to be expressed in terms of $k_m(x)$ and its derivatives. Second and later derivatives are related to $k_m(x)$ and $k_m'(x)$ by the differential equations

$$k_m''(x) = -\,k_m'(x)\left(\frac{m+4}{2x} + \frac{1}{x^2}\right) - \frac{(m+3)(m+1)}{16x^2}\,k_m(x),$$

$$k_m'''(x) = k_m'(x)\left\{\frac{3(m^2+12m+31)}{16x^2} + \frac{m+6}{x^3} + \frac{1}{x^4}\right\}$$

$$+ \frac{(m+3)(m+1)}{16}\,k_m(x)\left(\frac{m+8}{2x^3} + \frac{1}{x^4}\right),$$

$$k_m^{iv}(x) = -\,k_m'(x)\left\{\frac{m^3+24m^2+173m+360}{16x^3}\right.$$

$$\left. + \frac{5(m^2+16m+57)}{8x^4} + \frac{3(m+8)}{2x^5} + \frac{1}{x^6}\right\}$$

$$- \frac{(m+3)(m+1)}{16}\,k_m(x)\left\{\frac{3(m^2+20m+95)}{16x^4} + \frac{m+10}{x^5} + \frac{1}{x^6}\right\},$$

$$k_m^{v}(x) = k_m'(x) \left\{ \frac{5(m^4+40m^3+554m^2+3080m+5589)}{16^2 \; x^4} \right.$$

$$+ \frac{5(m^3+30m^2+275m+750)}{16\, x^5}$$

$$+ \frac{21(m^2+20m+91)}{16\, x^6}$$

$$\left. + \frac{2(m+10)}{x^7} + \frac{1}{x^8} \right\}$$

$$+ \frac{(m+3)(m+1)}{16} k_m(x) \left\{ \frac{m^3+36m^2+413m+1500}{16\, x^5} \right.$$

$$+ \frac{5(m^2+24m+137)}{8\, x^6}$$

$$\left. + \frac{3(m+12)}{2x^7} + \frac{1}{x^8} \right\},$$

$$k_m^{vi}(x) = -k_m'(x) \left\{ \frac{3(m^5+60m^4+1350m^3+14040m^2+66089m+108780)}{2\cdot 16^2 \; x^5} \right.$$

$$+\frac{35\,(m^4+48m^3+806m^2+552m+12681)}{16^2\,x^6}$$

$$+\frac{7\,(m^3+36m^2+401m+1356)}{8\,x^7}$$

$$\frac{9\,(m^2+24m+133)}{4x^8}+\frac{5\,(m+12)}{2x^9}+\frac{1}{x^{10}}\Bigg\}$$

$$\div\ \frac{(m+3)(m+1)}{16}\,k_m(x)$$

$$\times\Bigg\{\frac{5\,(m^4+56m^3+1130m^2+9688m+29589)}{16^2\,x^6}$$

$$+\ \frac{5\,(m^3+42m^2+563m+2394)}{16\,x^7}$$

$$+\ \frac{21\,(m^2+28m+187)}{16\,x^8}$$

$$+\ \frac{2(m+14)}{x^9}\ +\ \frac{1}{x^{10}}\Bigg\}, \qquad (2)$$

where primes denote differentiations with respect to x .

The uniform expansion for v then takes the form

$$v = k_m(x) \sum_{n=0}^{\infty} \frac{a_n(m,x)}{(8h)^n} + k_m'(x) \sum_{n=0}^{\infty} \frac{b_n(m,x)}{(8h)^{n+1}} . \qquad (3)$$

The first few coefficients are calculated to be

$$a_0 = 1 ,$$

$$a_1 = -\frac{1}{2^4}(m+1)(m+3) ,$$

$$a_2 = -\frac{1}{2^9}(m+1)(m+3) \left\{ (m^2+36m+23) - \frac{16(m^3+6m^2-5m+6)}{(m+1)(m+3)x} + \frac{16}{x^2} \right\} ,$$

$$a_3 = \frac{1}{3\cdot 2^{13}}\,(m+1)(m+3)$$

$$\times\Big[\,3\left(7m^4+24m^3-850m^2-1096m-1365\right)$$

$$-\frac{16}{x}\left\{\left(7m^3-18m^2-31m-18\right)-96\,\frac{(m-1)(m-2)(m-3)}{(m+1)(m+3)}\right\}$$

$$+\frac{48}{x^2}\left(m^2+12m+3\right)+\frac{256}{x^3}(m-4)-\frac{256}{x^4}\Big], \qquad (4a)$$

$$b_0 = \tfrac{1}{2}mx-1,$$

$$b_1 = -\frac{1}{2^5}\left\{\left(3m^3-4m^2+3m-12\right)x-2\left(3m^2+8m+3\right)\right.$$

$$\left.-\frac{8(m-4)}{x}+\frac{16}{x^2}\right\},$$

$$b_2 = \frac{1}{3\cdot 2^{10}}\left\{3\left(11m^5-104m^4+106m^3-368m^2+1499m-72\right)x\right.$$

$$+2\left(-25m^4+96m^3+250m^2+672m+207\right)$$

$$+\frac{32}{x}\left(-7m^3+24m^2+55m+36\right)$$

$$\left.+\frac{384}{x^2}\left(m^2+4m-7\right)+\frac{256}{x^3}(m-12)-\frac{512}{x^4}\right\}. \qquad (4b)$$

The uniform expansion for $y_2(z)$ is obtained by combining (8.3.5) and (3). The corresponding expansion for $y_1(z)$ follows from a combination of (8.3.4), (8.3.13) and (3); in this case Re in (1) is, of course, redundant.

Allowing for the slight difference between the definitions of k_m and k_a, values for $k_m(x)$ may be obtained with the aid of expressions and the list of tables given in § 7.51.

8.7 The special case of Mathieu functions of integer order.

In customary notation $ce(z)$ represents an even, and $se(z)$ an odd Mathieu function. Thus, since $y_1(z) = y_2(-z)$,

$$\begin{matrix} ce(z) \\ \\ se(z) \end{matrix} = N\left\{ y_1(z) \pm y_2(z) \right\}, \tag{1}$$

where the normalization factor N remains to be determined. The Mathieu functions of integer order and period π or 2π satisfy the conditions[+] (cf. Erdélyi et al. 1955, § 16.4)

$$
\left.
\begin{aligned}
&ce_{m_0}\left(z = \tfrac{1}{2}\pi\right) = 0\,, && \text{period } 2\pi,\\
&se_{m_0+1}\left(z = \tfrac{1}{2}\pi\right) = 0\,, && \text{period } \pi\\
&\left\{\frac{d}{dz}\, ce_{m_0-1}(z)\right\}_{z=\frac{1}{2}\pi} = 0\,, && \text{period } \pi\,,\\
&\left\{\frac{d}{dz}\, se_{m_0}(z)\right\}_{z=\frac{1}{2}\pi} = 0\,, && \text{period } 2\pi,
\end{aligned}
\right\}. \qquad (2)
$$

where m_0 is an odd integer. Combination of (2) with the expansions for $y_1\left(z \bumpeq \tfrac{1}{2}\pi\right)$ and $y_2\left(z \bumpeq \tfrac{1}{2}\pi\right)$ of § 8.5 leads to the following expansion for $m - m_0$, also obtained by Dingle and Müller (1962):

[+] The symmetry in q of the expansions of § 8.3 ensures that the conditions for ce and se at $z = 0$ are already complied with.

$$m - m_0 = \mp 2\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{(16h)^{\frac{1}{2}m_0} e^{-4h}}{\left\{\frac{1}{2}(m_0-1)\right\}!} \left[1 - \frac{3(m_0^2+1)}{2^6 h}\right.$$

$$+ \frac{1}{2^{13}h^2}\left(9m_0^4 - 40m_0^3 + 18m_0^2 - 136m_0 + 9\right)$$

$$- \frac{1}{2^{19}h^3}\left(9m_0^6 - 120m_0^5 + 467m_0^4 - 528m_0^3\right.$$

$$\left.\left. + 3307m_0^2 - 408m_0 + 1089\right) + O(h^{-4})\right], \quad (3)$$

with the upper sign for ce_{m_0} or ce_{m_0-1}, and the lower sign for se_{m_0+1} or se_{m_0}.

Both Goldstein (1927) and Ince (1932) adopt the normalizations

$$\int_{-\pi}^{\pi} \begin{matrix} ce^2 \\ se^2 \end{matrix}(z)\, dz = \pi, \quad (4)$$

except that Goldstein sets the particular normalization integral for $ce_0(z)$ equal to 2π. Ince's normalizations yield an expansion for N which is just that for $N_A\sqrt{2}$ given by (85) in Dingle and Müller (1962). Thus

$$N = \left(\frac{\pi}{8}\right)^{\frac{1}{4}} \frac{(16h)^{\frac{1}{4}m}}{e^{2h}\left\{\left[\frac{1}{2}(m-1)\right]!\right\}^{\frac{1}{2}}} \left\{ 1 - \frac{3(m^2+1)}{2^7 h} \right.$$

$$+ \frac{1}{2^{15}h^2}\left(7m^4 - 80m^3 - 26m^2 - 272m - 9\right)$$

$$- \frac{1}{2^{22}h^3}\left(3m^6 - 176m^5 + 1649m^4 + 1376m^3\right.$$

$$\left. + 12\,961\,m^2 + 2832m + 4275\right)$$

$$\left. + O\left(h^{-4}\right) \right\}. \qquad (5)$$

In N.B.S. (1951) the Mathieu function is normalized by fixing its value, or that of its derivative, at the point $z = 0$. In the notation of these tables the solutions are

$$\begin{matrix} Se_r(s,z) \\ \\ So_r(s,z) \end{matrix} = \begin{matrix} Ne \\ \\ No \end{matrix} \left\{ y_1(z) \pm y_2(z) \right\}, \qquad (6a,b)$$

where $s = 4h^2$, r integral. The normalizing conditions are

$$Se_r(s,0) = 1 \,, \quad \left\{ \frac{d}{dz} So_r(s,z) \right\}_{z=0} = 1. \tag{7}$$

Calculation with the aid of the expansions of § 8.3 yields

$$Ne = \frac{1}{2} \left\{ 1 - \frac{m}{2^4 h} - \frac{1}{2^{14} h^2} \left(m^4 + 214\, m^2 + 297 \right) - \frac{1}{2^{18} h^3} \left(3m^5 + 898\, m^3 + 5435\, m \right) + O(h^{-4}) \right\}, \tag{8a}$$

$$No = \frac{2\,Ne}{4h - m} \,. \tag{8b}$$

8.71 <u>Numerical calculations</u>.

The expansions of section 8.5 have the important advantage over those of Sips (1949) and of Dingle and Müller (1962) that they involve elementary functions, thus rendering them particularly suitable to calculation on a digital computer. We have programmed the expansions of §§ 8.5 and 8.7 for an IBM 1620 digital computer, thereby obtaining tables for the Mathieu functions $ce_{\frac{1}{2}(m-1)}(z,h)$ and $se_{\frac{1}{2}(m+1)}(z,h)$ in the range $z = 50^\circ(1^\circ)90^\circ$ for $m = 1(2)5$, $h = 5(0.1)10$, and for $m = 1(2)9$, $h = 5(5)85$.+ It is reasonable to take $m = m_0$ in this range of h since by (8.7.3) the difference between m and m_0 for $h = 5$ is less than 0.0001% of m_0. These tables will be deposited in the Royal Society Depository for unpublished tables.

The following is the programme used for the range $m = 1(2)5$, $h = 5(0.1)10$, written in Fortran language:

+ It will be noted that our tables start from the highest value of h considered in the National Bureau of Standards tables (1951).

```
C   VALUES OF MATHIEU FUNCTIONS CE(Z,H) AND SE(Z,H) OF INTEGER ORDER.
      DIMENSION TERM1(15),TERM2(7)
    1 FORMAT (10X,54HMATHIEU FUNCTIONS CE(Z,H) AND SE(Z,H) OF INTEGER OR
     1DER)
    2 FORMAT ( 7(2X,F14.9))
    3 FORMAT (20X,I2,2X,F14.9,2X,F14.9)
    4 FORMAT (1H1, 70X,6HPAGE  ,I3)
    5 FORMAT (////20X,4HH = ,F5.1,3H , ,4HM = ,I2)
    6 FORMAT (/20X,30H Z             CE              SE)
    7 FORMAT (2I3)
      DO 10 J=1, 7
      TERM1(J)=0.
      TERM2(J)=0.
   10 CONTINUE
      PI=3.141592653
      KOWNT=0
      SQRTHP=1.25331413731550
      IM=5
      DO 900 M=1,IM,2
   13 CONTINUE
      IF (SENSE SWITCH 2) 11,12
   11 READ 7,M,KOWNT
   12 CONTINUE
   16 H=4.9
   18 IH=50
      HC=.1
   17 XM=M
      XM1=(XM-1.)/4.
      M1=XM1
      XM2=M1
      MM=(M-1)/2
   41 IZZZ=1
   42 CONTINUE
      FACTN=1.
      IF (MM) 100,100,101
  101 CONTINUE
      DO 102 MMM=1,MM
      A=MMM
  102 FACTN=FACTN*A
  100 CONTINUE
      SQRTFC=1./SQRTF(FACTN)
      DO 800 IHH=1,IH
      H=H+HC
      BRAH1=1.
      BRAH2=3.*(XM*XM+1.)/(64.*H)
      IF (ABSF(BRAH1)-ABSF(BRAH2)) 21,21,23
   21 BRAHM=BRAH1
      GO TO 28
   23 CONTINUE
      BRAH3=((((9.*XM-40.)*XM+18.)*XM-136.)*XM+9.)/(64.*128.*H*H)
      IF (ABSF(BRAH2)-ABSF(BRAH3)) 24,24,25
   24 BRAHM=BRAH1-BRAH2
      GO TO 28
   25 BRAH4=((((((9.*XM-120.)*XM+467.)*XM-528.)*XM+3307.)*XM-408.)*XM+10
     189.)/(64.*64.*128.*H*H*H)
      IF (ABSF(BRAH3)-ABSF(BRAH4)) 26,26,27
   26 BRAHM=BRAH1-BRAH2+BRAH3
      GO TO 28
```

```
 27 BRAHM=BRAH1-BRAH2+BRAH3-BRAH4
 28 CONTINUE
    FACTM=1.
    IMD=(M-1)/2
    IF (IMD) 34,34,33
 33 CONTINUE
    DO 31 IID=1,IMD
    DII=IID
 31 FACTM=FACTM*DII
 34 BRAHM1=BRAHM/FACTM
    BRAHM2=2.*SQRTF(2./PI*16.*H**M)*EXPF(-4.*H)*BRAHM1
    IF (BRAHM2/XM-1.E-6) 39,39,800
 39 CONTINUE
    HM1=SQRTHP/(8.*H)**M1
    HM2=HM1*(8.*H)
    IF (XM1-XM2) 901,901,902
902 HM1=HM1/SQRTF(8.*H)
901 CONTINUE
    IF (XM1-XM2) 903,903,904
903 HM2=HM2/SQRTF(8.*H)
904 KOWNT=KOWNT+1
    PRINT 4,KOWNT
    PRINT 1
906 PRINT 5,H,M
    PRINT 6
    AMH=((PI/8.)*(16.*H)**M)**.25*EXPF(-2.*H)*(1.-3.*(XM*XM+1.)
   1/(128.*H)+(((((7.*XM-80.)*XM-26.)*XM-272.)*XM-9.)/(32768.*H*H)
   2-((((((3.*XM-176.)*XM+1649.)*XM+1376.)*XM+12961.)*XM+2832.)*XM
   3+4275.)/(4194304.*H**3))
    XNORM=AMH*SQRTFC
    DO 950 IZZ=50,90,IZZZ
    ZZ=IZZ
    Z=PI*ZZ/180.
    Q1=2./(1.+SINF(Z))-1.
    IF (XM1-XM2) 200,200,201
200 S=XM1
    GO TO 220
201 S=XM1-.5
220 R=-1.
202 R=R+1.
    IR=R
    IF (R-S-2.) 203,212,212
212 RES3=0.
    GO TO 401
203 CONTINUE
    IF(1.-R) 206,207,208
208 RES1=-1./(S+1.)
    GO TO 205
207 RES1=1.
    GO TO 205
206 PROD1=1.
    T=-1.
209 T=T+1.
    TS=-S+T
    IF (TS+1.) 210,210,205
210 PROD1=PROD1*TS
    RES1=PROD1
    IF (T-R+2.) 209,205,205
```

```
205 CONTINUE
    FACT3=1./1.7724538509
    IF (S) 215,215,216
216 IS=S
    DO 217 ISS=1,IS
    SS=ISS
    FACT3=FACT3*(-SS+.5)
217 CONTINUE
215 CONTINUE
    FACT4=FACT3*(-S-.5)
    IF (XM1-XM2) 279,279,280
280 XN=2.*R+1.
    GO TO 281
279 XN=2.*R
281 N=XN
    IFACT=1
    IF (N) 303,303,301
301 IN=0
304 IN=IN+1
    IFACT=IFACT*IN
    IF(N-IN) 303,303,304
303 FACT=IFACT
    FACT2=1./FACT
    IF (N) 390,390,391
390 FACT2A=1.
    GO TO 392
391 FACT2A=(32.*H*Q1)**IR
392 RES3=FACT2*FACT2A*RES1*FACT3
    IF (XM1-XM2) 401,401,402
402 RES3=-RES3*SQRTF(32.*H*Q1)*FACT4/FACT3
401 CONTINUE
    CALL BRAKET(XN,XM,H,BRA,BRA3)
    TERM1(IR+1)=RES3*(BRA*(.5*XN-.25*(XM+3.))+BRA3)
    IF (TERM1(IR+1))202,310,202
310 ACC1=0.
    IF (SENSE SWITCH 1) 700,701
700 PRINT 2,(TERM1(I1),I1=1, 7)
701 CONTINUE
    DO 320 K=1,IR
320 ACC1=ACC1+TERM1(K)
    SQRTQ=SQRTF(1.+Q1)
    EXPSIN=EXPF(2.*SINF(Z)*H)
    Y1=ACC1*HM1*SQRTQ*EXPSIN
    S1=XM1*2.
    S2=XM1
    IS2=S2
    S3=IS2
    R1=-1.
500 R1=R1+1.
    IF (S2-S3) 501,501,502
501 XN=2.*R1+1.
    PROD5=1.
    GO TO 503
502 XN=2.*R1
    S2=(S1-1.)/2.
    PROD5=S2+.5
503 CONTINUE
    IF (S2) 505,505,506
```

```
506 IS2=S2
    DO 507 ISS=1,IS2
    SSS=ISS
507 PROD5=PROD5*(SSS-.5)
505 FACT5=1./(1.7724538509*PROD5)
    IF (R1-1.) 520,520,508
508 IR1=R1
    DO 509 ISS=1,IR1
    SS=ISS
509 FACT5=FACT5*(S2+SS)
520 CONTINUE
    X=(XN+S1+1.)/2.
    X1=X/2.
    IX1=X1
    X2=IX1
    IF (X1-X2) 530,530,531
531 FACT5=-FACT5
530 PROD6=1.
    IF (XN) 540,540,541
541 IXN=XN
    DO 542 IXX=1,IXN
    XXN=IXX
542 PROD6=PROD6*XXN
540 FACT6=1./PROD6
    IR1=R1
    FACT7=1.
    IF (XN-1.) 543,544,576
576 FACT7=(32.*H*Q1)**IR1
    IF (XM1-S3) 544,544,543
544 FACT7=-FACT7*SQRTF(32.*H*Q1)
543 CONTINUE
    N=XN
    XM=-XM
    H=-H
    CALL BRAKET(XN,XM,H,BRA,BRA3)
595 CONTINUE
    IF (XM+3.) 591,590,590
590 BRAT2=0.
    GO TO 592
591 BRAT2=BRA3/(.5*XN-.25*(XM+3.))
592 TERM2(IR1+1)=FACT5*FACT6*FACT7*(BRA+BRAT2)
    XM=-XM
    H=-H
    IF (IR1) 500,500,587
587 CONTINUE
    IF (ABSF(TERM2(IR1+1))-ABSF(TERM2(IR1))) 585,551,551
585 CONTINUE
    IF (N-11) 500,551,551
551 CONTINUE
    IF (SENSE SWITCH 1) 702,703
702 PRINT 2,(TERM2(I2),I2=1, 7)
703 ACC2=0.
    DO 580 K=1,IR1
580 ACC2=ACC2+TERM2(K)
    Y2=ACC2*HM2*SQRTQ/EXPSIN
    CE=(Y1+Y2)*XNORM
    SE=(Y1-Y2)*XNORM
    PRINT 3,IZZ,CE,SE
```

```
950 CONTINUE
800 CONTINUE
 20 CONTINUE
900 CONTINUE
    STOP
    END
```

```
 SUBROUTINE BRAKET(XN,XM,H,BRA,BRA3)
 BRA1=                      (1.-(4.*XN**2+4.*(XM-1.)*XN-3.*(XM-1.)
1**2)/(128.*H)
1+(16.*XN**4+32.*(XM-3.)*XN**3-8.*(XM*XM+10.*XM-19.)*XN*XN-8.*(3.*
4XM**3+XM*XM-11.*XM+15.)*XN+(9.*XM**4+36.*XM**3-178.*XM*XM+244.*XM-
5111.))/(32.*1024.*H*H))
 BRA2=                  1./(12.*1048576.*H**3)*(64.*XN**6+192.*(XM-5
1.)*XN**5+16.*(3.*XM*XM-126.*XM+331.)*XN**4-32.*(7.*XM**3+27.*XM*XM
2-259.*XM+441.)*XN**3-4.*(3.*XM**4-276.*XM**3-654.*XM*XM+3180.*XM
3-4429.)*XN*XN+4.*(33.*XM**5+315.*XM**4-94.*XM**3-498.*XM*XM+3949.*
4XM-2937.)*XN)
 BRA4=                  1./(12.*1048576.*H**3)
1*3.*(15.*XM**6+206.*XM**5+545.*XM**4-4220.*XM**3+10561.*XM*XM
2-11250.*XM+4143.)
 BRA3=(.5*(XM+3.)*(XM+3.)*(XM+1.)*(XM+1.)*(2.*XN-XM+1.))/
1(1024.*32.*H*H)+(XM+3.)*(XM+3.)*(XM+1.)*(XM+1.)*(XM+2.)*(2.*XN-XM
2+1.)/(16.*1048576.*H**3)
 BRA=BRA1-BRA2-BRA4
 RETURN
 END
```

To do justice to the high accuracy of our expansions (see e.g. Table 2 below) when m is small and h is large, the field-length has been fixed at 10 digits.

Table 1 illustrates the minimum accuracy of the tables for the range $m = 1(2)5$. Table 2 indicates the accuracy achieved for $m=1$ and $h=40$ in comparison with results based on Sips's expansions (1949). For $h=5$, the tables for $m=7$ and $m=9$ are accurate to at least two significant figures, the accuracy increasing rapidly as h gets larger. The reference values were taken from the National Bureau of Standards tables (1951).

TABLE 1.

	$ce_0(z,5)$		$ce_1(z,5)$	
z	approx. value	N.B.S. value	approx. value	N.B.S. value
50°	0.170271	0.170293	0.5331	0.5336
60°	0.449844	0.449875	1.0354	1.0359
70°	0.921224	0.921248	1.3939	1.3943
80°	1.429478	1.429480	1.0726	1.0727
90°	1.657521	1.657510	0.0000	0

TABLE 1. (continued)

	$se_3(z,5)$	
z	approx. value	N.B.S. value
50°	1.008	1.014
60°	1.299	1.304
70°	0.798	0.800
80°	-0.419	-0.421
90°	-1.113	-1.116

TABLE 2.

	$ce_0(z,40)$	
z	approx. value	N.B.S. value
55°	0.000001535	0.00000154
60°	0.000064473	0.0000645
65°	0.001600602	0.0016005
75°	0.185745234	0.18575
90°	2.812111528	2.812111

9. ASYMPTOTIC EXPANSIONS FOR OBLATE SPHEROIDAL WAVE FUNCTIONS.

9.1 Introduction.

The techniques of earlier chapters are illustrated further by calculations for the angular oblate spheroidal wave functions. These functions arise, for instance, when the Schrödinger equation is separated in oblate spheroidal co-ordinates. (Separation in prolate spheroidal co-ordinates yields correspondingly the equations satisfied by the prolate spheroidal wave functions.)

The oblate spheroidal co-ordinate system is formed by rotating the two-dimensional elliptic co-ordinate system about the minor axis of the ellipse. The oblate spheroidal co-ordinates ξ , η and ϕ are related to the Cartesian co-ordinates x , y and z by the equations

$$\left.\begin{aligned} x &= c\left\{(1+\xi^2)(1-\eta^2)\right\}^{\frac{1}{2}}\cos\phi, \\ y &= c\left\{(1+\xi^2)(1-\eta^2)\right\}^{\frac{1}{2}}\sin\phi, \\ z &= c\,\xi\,\eta, \end{aligned}\right\} \qquad (1)$$

where $2c$ is the interfocal distance along the z-axis, the axis of rotation. The variables are restricted to the ranges

$$0 \leq \xi \leq \infty, \quad -1 \leq \eta \leq 1, \quad 0 \leq \phi \leq 2\pi. \tag{2}$$

When the wave equation,

$$\nabla^2 \psi + k^2 \psi = 0, \tag{3}$$

is separated in the oblate spheroidal co-ordinate system by assuming solutions of the form

$$\psi = R(\xi)\, A(\eta)\, \Phi(\phi), \tag{4}$$

we obtain the equations

$$\frac{d}{d\xi}\left[(1+\xi^2)\frac{d}{d\xi}R(\xi)\right] - \left\{\lambda - 4h^2(1+\xi^2) - \frac{r^2}{1+\xi^2}\right\}R(\xi) = 0, \tag{5a}$$

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{d}{d\eta}A(\eta)\right] + \left\{\lambda - 4h^2(1-\eta^2) - \frac{r^2}{1-\eta^2}\right\}A(\eta) = 0, \tag{5b}$$

$$\frac{d^2}{d\phi^2}\,\Phi(\phi) + r^2\,\Phi(\phi) = 0, \tag{5c}$$

where $4h^2 = k^2 c^2$, and λ and r denote the separation constants. In the following, we will only consider functions ψ which are single-valued and periodic. Since

$$\Phi(\phi) \propto e^{\pm i r \phi}, \tag{6}$$

this means that we restrict r to be an integer.

The equations for the angle function $A(\eta)$ and the radial function $R(\xi)$ are related by the simple transformation $\xi = \pm i\eta$ but since these variables have different ranges, $A(\eta)$ and $R(\xi)$ cannot easily be treated simultaneously. However, the calculations for these functions are not fundamentally different and attention will be confined to (5b). Those solutions of (5b) which are finite and single-valued will be termed oblate spheroidal wave functions. The values of λ belonging to these functions are then said to be their eigenvalues.

9.2 Brief survey of the history and applications of spheroidal wave functions.

Spheroidal wave functions were first investigated by Niven (1880) in connexion with the conduction of heat in spheroidal bodies. As the eccentricity of the spheroid goes to zero, the problem becomes soluble in terms of spherical harmonics. Niven was thus led to express the spheroidal angle functions as a series involving Legendre functions. Maclaurin (1898) considered several applications of spheroidal wave functions and expressed them as power series expansions. Integral equations were derived by Abraham (1899) and later by Poole (1923). Möglich (1927) considered the separation of the wave equation in general ellipsoidal co-ordinates and transformed the resultant differential equations into homogeneous integral equations. He obtained integral equations for spheroidal wave functions as a special case of his results. More recently, spheroidal wave functions and their eigenvalues have been studied among others by Bouwkamp (1941, 1947)+, Chu and Stratton (1941), Meixner (1947, 1948) Sips (1949), Abramo-

+ The method given here was developed independently by Blanch (1946).

witz (1949) and Müller (1960).

The following studies may be mentioned to illustrate the importance of oblate spheroidal wave functions in the fields of physics and engineering: antenna theory (Leitner and Spence 1950); acoustic diffraction by circular disks and apertures (Kotani 1933, Bouwkamp 1947, Meixner and Fritze 1949); sound waves from a freely vibrating circular disk (Gutin 1937, Meixner and Fritze 1949, Bouwkamp 1950a); energy levels of nucleon moving in potential well of oblate spheroidal shape (Granger and Spence 1951) based on Rainwater's (1950) spheroidal nuclear model; electromagnetic diffraction by circular disks and apertures (Meixner and Andrejewski 1950, Bouwkamp 1950b, Flammer 1953); electromagnetic scattering by spheroids (Rauch 1953).

Further accounts of, and references to, studies on the mathematical and applied aspects of spheroidal wave functions may be found in the tract by Strutt (1932), in the text-book by Meixner and Schäfke (1954), in Flammer's monograph on spheroidal wave functions (1957) and in vol. II of 'Methods of theoretical physics' by Morse and Feshbach (1953).

Tables of spheroidal wave functions are as yet incomplete. Leitner and Spence (1950) have computed values of some oblate spheroidal wave functions. Flammer (1957) also gives some tables for spheroidal wave functions. The most extensive tables available to date are those of Stratton, Morse, Chu, Little and Corbató (1956) giving expansion coefficients and eigenvalues.

9.3 The integro-differential equation.

The equation for $A(\eta)$ given by (9.1.5b) is brought into a form suitable for the application of our methods by changing the independent and dependent variables according to

$$\eta = \cos z ,$$

$$A(\cos z) = (\sin^2 z)^{-\frac{1}{4}} \, y(z) .$$

Under these transformations (9.1.5b) becomes

$$\frac{d^2 y}{dz^2} = \left\{ 4 h^2 \sin^2 z + \frac{r^2 - \frac{1}{4}}{\sin^2 z} - \lambda - \frac{1}{4} \right\} y , \qquad (1)$$

where h^2 is large and positive and λ represents the characteristic numbers. The form of the asymptotic expansion for λ when h^2 is large is well-known (cf. e.g. Flammer 1957, § 8.2.1), and for later convenience we write

$$\lambda = 2hm - \frac{1}{8}\left(m^2 - 4r^2 + 4\right) + \sum_{j=0}^{\infty} \frac{a_j}{(8h)^j}, \tag{2}$$

where m is approximately an even integer, and a_j are coefficients depending only on m. (We will show that $a_0 = 0$ so that the second term in (2) is in fact the term of $O(h^0)$ in the expansion for λ.) By (2), when h^2 is large the leading terms on the right-hand side of (1) are $(4h^2 \sin^2 z - 2hm)y$, the first of these being dominant except where $|z| \doteq 0$ or π. Excluding these regions for the present, we obtain as zero-order approximations to a pair of independent solutions of (1)

$$y^0_{\pm} \propto \left(\sin^2 z\right)^{-\frac{1}{4}} \exp\left[\pm \int^z \left(4h^2 \sin^2 z\right)^{\frac{1}{2}} \left\{1 - \frac{m}{4h\sin^2 z}\right\} dz\right], \tag{3}$$

and so, noting that (1) remains unchanged on replacing z with $z - \pi$ or $\pi - z$, we write a pair of solutions as

$$y_1(z) = (\sin^2 z)^{-\frac{1}{4}} e^{2h\cos z} \left(\frac{1+\cos z}{1-\cos z}\right)^{-\frac{1}{4}m} v(z-\pi), \tag{4}$$

$$y_2(z) = (\sin^2 z)^{-\frac{1}{4}} e^{-2h\cos z} \left(\frac{1+\cos z}{1-\cos z}\right)^{\frac{1}{4}m} v(z), \tag{5}$$

where $v = 1 + O(h^{-1})$. Henceforth it is therefore only necessary to consider one of these solutions, y_2 say, in detail.

The simplifying variable for the differential equation satisfied by v suggested by analogy with the calculations for Mathieu functions is

$$q = \frac{1 + \cos z}{1 - \cos z}. \tag{6}$$

With this variable, and λ expressed as in (2), substitution for v into (1) yields the integro-differential equation

$$8hv = (q+1)^2 v' + \frac{m}{2q}(q+1)^2 v$$

$$+ \frac{1}{16}\int^q \left\{ \frac{(m+2-2r)(m+2+2r)}{q^2} + (m-2-2r)(m-2+2r) \right\} v\,dq$$

$$+ \sum_{j=0}^{\infty} \frac{a_j}{(8h)^j} \int^q \frac{v}{q}\,dq\,, \qquad (7)$$

where the prime signifies differentiation with respect to q . (It will be noted that the corresponding equation for Mathieu functions is recovered on taking $r^2 = \frac{1}{4}$.)

An expansion for v in negative powers of h is obtained on solving (7) by an iterative procedure. Denoting by v_i the ith order term in this expansion, the next higher-order term v_{i+1} satisfies the recurrence relation:

$$8hv_{i+1} = (q+1)^2 v_i' + \frac{m}{2q}(q+1)^2 v_i$$

$$+ \frac{1}{16}\int^q \left\{ \frac{(m+2-2r)(m+2+2r)}{q^2} + (m-2-2r)(m-2+2r) \right\} v_i\,dq$$

$$+ \sum_{j=0}^{i} \frac{a_j}{(8h)^j} \int^q \frac{1}{q} v_{i-j}\,dq\,. \qquad (8)$$

The simplest possible expansion for v is obtained by choosing the lower limits of integration in (8) such that no constant terms appear in $v_{i \neq 0}$. With $v_0 = 1$, (8) then yields

$$v_1 = \frac{1}{2^7 h}\left\{(m+2-2r)(m+2+2r)q - (m-2+2r)(m-2-2r)\frac{1}{q}\right\} + \frac{1}{8h} a_0 \ln q . \tag{9}$$

We require that the solutions be unique as well as periodic so that $a_0 = 0$. In general, the coefficient a_{i-1} in (2) is obtained by setting the coefficient of the logarithmic term in v_i equal to zero. Substitution for v_1 into (8) yields

$$\begin{aligned} v_2 = \frac{1}{2^{15} h^2}\Big\{ & (m+2+2r)(m+2-2r)(m+6+2r)(m+6-2r)\, q^2 \\ & + 32\,(m+2)(m+2-2r)(m+2+2r)\, q \\ & - 32(m-2)(m-2+2r)(m-2-2r)\frac{1}{q} \\ & + (m-2-2r)(m-2+2r)(m-6-2r)(m-6+2r)\frac{1}{q^2}\Big\}, \end{aligned} \tag{10a}$$

where the coefficient of the logarithmic term vanishes for

$$a_1 = -\frac{m}{16}\left(m^2 - 4r^2 + 4\right). \tag{10b}$$

Similarly,

$$v_3 = \frac{1}{3\cdot 2^{22} h^3}\Big\{(m+2+2r)(m+2-2r)(m+6+2r)(m+6-2r)$$

$$\times (m+10+2r)(m+10-2r)\, q^3$$

$$+ 96\,(m+2+2r)(m+2-2r)(m+6+2r)(m+6-2r)(m+3)\, q^2$$

$$+ 3\,(m+2+2r)(m+2-2r)\Big\{m^4 - 8m^3 + 728m^2$$

$$+ 2784m + 3344 - 8(m^2 - 4m + 36)r^2 + 16r^4\Big\}\, q$$

$$- 3\,(m-2+2r)(m-2-2r)\Big\{m^4 + 8m^3 + 728m^2 - 2784m$$

$$+ 3344 - 8(m^2 + 4m + 36)r^2 + 16r^4\Big\}\frac{1}{q}$$

$$+ 96\,(m-2+2r)(m-2-2r)(m-6+2r)(m-6-2r)(m-3)\frac{1}{q^2}$$

$$- (m-2+2r)(m-2-2r)(m-6+2r)(m-6-2r)$$

$$\times \left(m-10+2r\right)\left(m-10-2r\right)\frac{1}{q^3}\Big\}\,, \tag{11a}$$

$$a_2 = -\frac{1}{2^6}\left\{\left(5m^4+40m^2+16\right)-8r^2\left(3m^2+4\right)+16r^4\right\}. \tag{11b}$$

The transformation $z \to \pi - z$ is by (6) equivalent to $q \to 1/q$, which in (7) corresponds to $(m,h) \to (-m,-h)$. Consequently, the second solution is given by

$$V(\pi-z) = V(q,-m,-h). \tag{12}$$

The above series is useful provided

$$|h^{-1}| << |q| << |h|, \tag{13}$$

but breaks down when q is either very large or very small, i.e. when $|z| \sim 0$ or π. A similar expansion has been obtained by Müller (1960) with the aid of a perturbation technique.

9.4 <u>The Mellin transform of V</u>.

To obtain expansions of wider applicability than those of the previous section, we now solve (9.3.7) by means of the Mellin transform.

Although the oblate spheroidal wave functions with which we are concerned here are real for real z , it is convenient to carry the calculations through for a general solution of (9.3.7), V say, of which v is the real part. Let $M(\mu)$, the Mellin transform of V , be defined by

$$V = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M(\mu)\left(-\frac{q}{8h}\right)^{-\mu} d\mu . \qquad (1)$$

Substitution for V into (9.3.7) yields the difference equation

$$\left\{\mu - \tfrac{1}{4}(m-2-2r)\right\}\left\{\mu - \tfrac{1}{4}(m-2+2r)\right\} M(\mu+1) - \mu M(\mu)$$

$$= \frac{M(\mu)}{8h}\,\mu(2\mu-m) - \frac{M(\mu-1)}{(8h)^2}\left\{\mu-\tfrac{1}{4}(m+2-2r)\right\}\left\{\mu-\tfrac{1}{4}(m+2+2r)\right\}$$

$$+ \sum_{j=0}^{\infty} \frac{a_j}{(8h)^j} , \qquad (2)$$

which is in the right form for solving by successive approximations.

With $h \to \infty$, the zero-order approximation $M_0(\mu)$ is seen to satisfy

$$\left\{\mu - \tfrac{1}{4}(m-2-2r)\right\}\left\{\mu - \tfrac{1}{4}(m-2+2r)\right\} M_0(\mu+1) - \mu M_0(\mu) = 0. \quad (3)$$

From the possible solutions it appears, as will be shown later (cf. § 9.5), that for integer r we must select

$$M_0(\mu) = (-1)^{\mu} \frac{(\mu-1)!\left\{-\mu + \tfrac{1}{4}(m-2+2r)\right\}!}{\left\{\mu - \tfrac{1}{4}(m+2-2r)\right\}!}. \quad (4)$$

If now a function $m(\mu)$ is defined by

$$M(\mu) = M_0(\mu)\, m(\mu), \quad (5)$$

successive contributions to $m(\mu)$ are given by the recurrence relation

$$m_{i+1}(\mu) = \frac{1}{8h} \int (2\mu - m)\, m_i(\mu)$$

$$- \frac{1}{(8h)^2} \mathcal{S} \frac{\left\{\mu - \frac{1}{4}(m+2-2r)\right\}^2 \left\{\mu - \frac{1}{4}(m+2+2r)\right\}^2}{\mu(\mu-1)} m_{i-1}(\mu-1)$$

$$+ \sum_{j=0}^{i} \frac{a_j}{(8h)^{j+1}} \mathcal{S} \frac{m_{i-j}(\mu)}{\mu}. \qquad (6)$$

The initial constant term $m_0(\mu)$ and the summation constants will be chosen so as to achieve agreement with the expansion for v given in § 9.3. Therefore $m_0(\mu)$ is taken equal to

$$\frac{\left\{\frac{1}{4}(2r-m-2)\right\}!}{\left\{\frac{1}{4}(2r+m-2)\right\}!}$$

Summation in (6) then yields

$$m_1(\mu) = \frac{\left\{\frac{1}{4}(2r-m-2)\right\}!}{8h\left\{\frac{1}{4}(2r+m-2)\right\}!} \left\{\mu(\mu-m-1) + a_0\,\psi(\mu-1)\right\}, \quad (7)$$

where $\psi(\mu) = (d/d\mu)\ln(\mu!)$. Retention of $\psi(\mu-1)$ in $m_1(\mu)$ would lead to double and higher-order poles in $M(\mu)$, and consequently to logarithmic terms in expansions of the type derived in § 9.3. As already stated, such terms would not give unique solutions. Accordingly $a_0 = 0$, as in § 9.3. Continuing the iteration, and again equating the coefficient of $\psi(\mu-1)$ to zero, we obtain

$$m_2(\mu) = \frac{\left\{\frac{1}{4}(2r-m-2)\right\}!}{(8h)^2\left\{\frac{1}{4}(2r+m-2)\right\}!}\left\{\frac{1}{2}\mu^4 - (m+2)\mu^3 + \frac{1}{2}(m^2+5m+5)\mu^2 - \frac{1}{8}(7m^2+16m-4r^2+12)\mu + \frac{(m-3)^2(m-1)^2 - (4r^2-1)(2m^2-8m+7-4r^2)}{16^2}\frac{\mu}{\mu-1}\right\}, \tag{8}$$

with $a_1 = -\frac{m}{16}(m^2-4r^2+4)$, as in (9.3.10b).

Calculation of the third-order term yields a_2 as in (9.3.11b). Thus by (5), to $O(h^{-3})$,

$$M(\mu) = \frac{\left\{\frac{1}{4}(2r-m-2)\right\}!}{\left\{\frac{1}{4}(2r+m-2)\right\}!}(-1)^{\mu}\frac{\left\{-\mu+\frac{1}{4}(m-2+2r)\right\}!}{\left\{\mu-\frac{1}{4}(m+2-2r)\right\}!}$$

$$\times \Bigg[(\mu-1)! + \frac{1}{8h} (\mu - m - 1)\mu!$$

$$+ \frac{1}{(8h)^2} \left\{ \frac{1}{2}\mu^3 - (m+2)\mu^2 + \frac{1}{2}(m^2+5m+5)\mu \right.$$

$$\left. - \frac{1}{8}(7m^2+16m-4r^2+12) \right\} \mu!$$

$$+ \frac{1}{(8h)^2 16^2} \left\{ (m-3)^2(m-1)^2 - (4r^2-1)(2m^2-8m+7-4r^2) \right\} \mu(\mu-2)!$$

$$+ \frac{1}{768(8h)^3} \left\{ 128\mu^5 - 384(m+3)\mu^4 + 128(3m^2+21m+31)\mu^3 \right.$$

$$- 32(4m^3+69m^2+224m+210+12r^2)\mu^2$$

$$+ \Big(3m^4+648m^3+4386m^2+8664m+6011$$

$$- 3(4r^2-1)(2m^2+24m+103-4r^2) \Big)\mu$$

$$- \Big(3m^5 - 24m^4 + 994m^3 + 3384m^2 + 5243m + 2688$$

$$\left. - 3(4r^2-1)(2m^3-8m^2+96m+135-4mr^2) \Big) \right\} \mu!$$

$$+\frac{(m-2)}{256(8h)^3}\left\{(m-3)^2(m-1)^2-(4r^2-1)(2m^2-8m+7-4r^2)\right\}$$

$$\times\ \mu(\mu-2)!$$

$$\left. + O(h^{-4})\right]. \qquad (9)$$

9.5 Expansions for $z \simeq 0$ and $z \simeq \pi$.

The separation constant c in the Mellin-Barnes integrals for V is chosen such that the contour line from $c-i\infty$ to $c+i\infty$ separates the poles of $\left\{-\mu+\frac{1}{4}(2r+m-2)\right\}!$ from those of the factorial functions in square brackets in (9.4.9). When the contour of integration is closed in an anti-clockwise sense, calculation of the residues at the enclosed poles leads to the expansions already obtained in §9.3.

When the contour is closed in a clockwise sense, calculation of the residues at the enclosed poles yields expansions which are valid for $z \simeq 0$, one of the regions where the expansions of §9.3 break down. The expression for $M(\mu)$ shows that V is real in this case, and hence by (9.4.1)

$$v(z \to 0) = \frac{\left\{\frac{1}{4}(2r-m-2)\right\}!}{\left\{\frac{1}{4}(2r+m-2)\right\}!}\left(\frac{q}{8h}\right)^{-a}$$

$$\times \sum_{n=0}^{\infty}\left(-\frac{8h}{q}\right)^{n}\frac{1}{n!\,(n+r)!}\Big[(n+a-1)!$$

$$+\frac{1}{8h}(n+a-m-1)(n+a)!$$

$$+\frac{1}{(8h)^2}\Big\{\frac{1}{2}(n+a)^3-(m+2)(n+a)^2$$

$$+\frac{1}{2}(m^2+5m+5)(n+a)$$

$$-\frac{1}{8}(7m^2+16m-4r^2+12)\Big\}(n+a)!$$

$$+\frac{(n+a)}{(8h)^2}\left\{\frac{(m-3)^2(m-1)^2-(4r^2-1)(2m^2-8m+7-4r^2)}{16^2}\right\}(n+a-2)!$$

$$+\frac{1}{768(8h)^3}\Big\{128(n+a)^5-384(m+3)(n+a)^4$$

$$+128(3m^2+21m+31)(n+a)^3$$

$$-32(4m^3+69m^2+224m+210+12r^2)(n+a)^2$$

$$+\Big(8m^4+648m^3+4386m^2+8664m+6011$$

$$-3(4r^2-1)(2m^2+24m+103-4r^2)\Big)(n+a)$$

$$-\Big(3m^5-24m^4+994m^3+3384m^2+5243m+2688$$

$$-3(4r^2-1)(2m^3-8m^2+96m+135-4mr^2)\Big)\Big\}(n+a)!$$

$$+\frac{(n+a)}{256(8h)^3}(m-2)\Big\{(m-3)^2(m-1)^2-(4r^2-1)(2m^2-8m+7-4r^2)\Big\}$$

$$\times(n+a-2)!$$

$$+O(h^{-4})\Big], \text{ where } \quad a=\tfrac{1}{2}r+\tfrac{1}{4}m+\tfrac{1}{2}. \qquad (1)$$

The corresponding expression for $V(z-\pi)$ is obtained simply by reversing the signs of m and h in (1) and taking the real part.

The case $|z| \doteq \pi$ need not be considered separately, since (9.3.4) and (9.3.5) show that

$$y_1(z+\pi)=y_2(z) \quad \text{or} \quad y_2(z-\pi)=y_1(z) \qquad (2)$$

The expansions of this section have the advantage over comparable expansions obtained by Meixner (1948), Sips (1949) and Müller (1960) in terms of Laguerre functions in that they involve only elementary functions. They are therefore particularly well-suited to programming for a digital computer.

It can now be seen why the alternative solution of (9.4.3), namely

$$M_0(\mu) = (\mu-1)!\ \left\{-\mu + \tfrac{1}{4}(m-2+2r)\right\}!\left\{-\mu + \tfrac{1}{4}(m-2-2r)\right\}!, \qquad (3)$$

had to be rejected. The expansion for ν for large q would then have included contributions from the residues of the poles of $\left\{-\mu+\frac{1}{4}(m-2+2r)\right\}!\left\{-\mu+\frac{1}{4}(m-2-2r)\right\}!$. The difference between the arguments of these factorial functions is r, and it is evident that when r is an integer, double poles would have resulted thus leading to logarithmic terms in the expansion for ν. This solution had therefore to be discarded, as such logarithmic terms violate the requirement of unique solutions. Incidentally, (3) is the required solution of (9.4.3) when r is non-integral. The results for the Mathieu functions are then recovered on setting $r = \pm\frac{1}{2}$.

9.6 Uniform expansions.

The expansions given in §§ 9.3 and 9.5 between them cover practically the whole range of z. The uniform expansion, including these expansions as special cases, is obtained by expressing the Mellin-Barnes integrals which occur in (9.4.1) in terms of a subsidiary function and its first derivative. As in previous applications of the Mellin transform method, a suitable function, $k_m^r(x)$ say, is defined by the inverse Mellin transform of the first term in (9.4.9). Confining our attention only to real functions, we therefore take

$$k_m^r(x) = \mathrm{Re}\ \frac{\{\frac{1}{4}(2r-m-2)\}!}{\{\frac{1}{4}(2r+m-2)\}!}$$

$$\times \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (\mu-1)!\ \frac{\{-\mu+\frac{1}{4}(m-2+2r)\}!}{\{\mu-\frac{1}{4}(m+2-2r)\}!}\,(-x)^{-\mu}\,d\mu, \quad (1)$$

where $-x = q/8h$, and Re signifies 'real part'. This function is essentially a generalised Laguerre function, and in fact (cf. Slater 1960, § 5.5)

$$k_m^r(x) = \operatorname{Re}(-1)^{\frac{1}{4}(m+2+2r)} \left\{-\tfrac{1}{4}(m+2+2r\right\}! \\ \times L^r_{-\frac{1}{4}(m+2+2r)}\left(\tfrac{1}{x}\right). \qquad (2)$$

The remaining Mellin-Barnes integrals can now be expressed in terms of $k_m^r(x)$ and its derivatives by the techniques of earlier sections. Moreover, second and higher-order derivatives are related to $k_m^r(x)$ and its first derivative by the differential equation

$$k_m^{r\,''}(x) = -\frac{1}{2x^2}\left[\left\{(m+4)x+2\right\}k_m^{r\,'}(x) \right. \\ \left. + \tfrac{1}{8}(m+2-2r)(m+2+2r)\, k_m^r(x)\right], \quad (3)$$

primes denoting differentiations with respect to x .

The amount of labour involved in calculating the terms in the uniform expansion may be reduced by making use of the results for Mathieu functions. We write (3) as

$$k_m^{r\,''}(x) = -\frac{1}{2x^2}\left[\left\{(m+4)x+2\right\} k_m^{r\,'}(x)\right.$$

$$+ \frac{1}{8}(m+1)(m+3)\, k_m^r(x) + \frac{(4r^2-1)}{16x^2}\, k_m^r(x)$$

$$= {}^r k_m'' + \frac{4r^2-1}{16x^2}\, k_m^r\,, \qquad (4)$$

so that the expression for ${}^r k_m''$ corresponds in form to that for k_m in the case of the Mathieu functions (cf. (8.6.2)). Later derivatives of k_m^r are now found by adding the extra contributions due to the term involving k_m^r in (4) to the corresponding terms already calculated for Mathieu functions. For instance, the third derivative is given by

$$k_m^{r\,'''}(x) = {}^r k_m''' - \frac{4r^2-1}{8\ x^3}\, k_m^r + \frac{4r^2-1}{16x^2}\, k_m^{r\,'}\,, \qquad (5)$$

where ${}^r k_m'''$ follows from the form of (8.6.2), and (4).

After some tedious algebra, the uniform expansion for v finally takes the form

$$v = k_m^r(x) \sum_{n=0}^{\infty} \frac{a_n(m,r,x)}{(8h)^n}$$

$$+ k_m^{r\,'}(x) \sum_{n=0}^{\infty} \frac{b_n(m,r,x)}{(8h)^{n+1}}\,, \qquad (6)$$

where for the first few coefficients

$$a_0 = 1 ,$$

$$a_1 = -\frac{1}{2^4}(m+2+2r)(m+2-2r) ,$$

$$a_2 = -\frac{(m+2+2r)(m+2-2r)}{2^9}\Big(3m^2+28m+28+4r^2$$

$$-\frac{16m}{x}+\frac{16}{x^2}\Big)$$

$$+\frac{(m-3)^2(m-1)^2-(4r^2-1)(2m^2-8m+7-4r^2)}{2^8(m-2+2r)(m-2-2r)}$$

$$\times\left\{(m+2+2r)(m+2-2r)+\frac{16}{x}\right\} ,$$

$$a_3 = \frac{1}{3\cdot 2^{13}}\Big[(m+3)(m+1)$$

$$\times\Big\{3(7m^4-8m^3-658m^2-1448m-1173)$$

$$-\frac{16}{x}(7m^3-18m^2-31m-18)$$

$$+ \frac{48}{x^2}(m^2+12m+3) + \frac{256}{x^3}(m-4) - \frac{256}{x^4} \Bigg\}$$

$$+(4r^2-1)\Bigg\{ (-19m^4-56m^3+1807m^2+4700m+4227$$

$$-36m^2r^2+240mr^2-1396r^2-80r^4)$$

$$+\frac{16}{x}(10m^3-12m^2-43m-132-12mr^2+24r^2)$$

$$-\frac{16}{x^2}(6m^2+48m+1461-12r^2)$$

$$-\frac{256}{x^3}(m-4)+\frac{256}{x^4}\Bigg\}$$

$$+96(m-2)\left\{\frac{(m-3)^2(m-1)^2-(4r^2-1)(2m^2-8m+7-4r^2)}{(m-2+2r)(m-2-2r)}\right\}$$

$$\times\left\{(m+2+2r)(m+2-2r)+\frac{16}{x}\right\}\Bigg], \qquad (7a)$$

$$b_0 = \tfrac{1}{2}(mx-1),$$

$$b_1 = \frac{1}{2^5}\Bigg\{(-3m^3+2m^2+5m+6)x+2(3m^2+8m+3)$$

$$+\frac{8}{x}(m-4)-\frac{16}{x^2}$$

$$+ (4r^2-1)\left((m-2)x-2\right)$$

$$+2\,\frac{(m-3)^2(m-1)^2-(4r^2-1)(2m^2-8m+7-4r^2)}{(m-2+2r)(m-2-2r)}\,x\Bigg\},$$

$$b_2 = \frac{1}{3\cdot 2^{10}}\Bigg[3x(11m^5-104m^4+42m^3+16m^2$$

$$+795m+312)$$

$$+2(-25m^4+96m^3+250m^2+672m+207)$$

$$+\frac{32}{x}(-7m^3+24m^2+55m+36)$$

$$+\frac{384}{x^2}(m^2+4m-7)+\frac{256}{x^3}(m-12)-\frac{512}{x^4}$$

$$+(4r^2-1)\Big\{x(-18m^3+112m^2+121m+148+20mr^2)$$

$$+2(10m^2+64m+2527-16r^2)+\frac{32}{x}(3m-4)-\frac{128}{x^2}\Big\}$$

$$+192(m-2)x\left\{\frac{(m-3)^2(m-1)^2-(4r^2-1)(2m^2-8m+7-4r^2)}{(m-2+2r)(m-2-2r)}\right\}\Bigg].\quad(7b)$$

The uniform expansion for $y_2(z)$ is now obtained by combining (9.3.5) and (6). The corresponding expansion for $y_1(z)$ follows from a combination of (9.3.4), (9.3.12) and (6).

9.7 <u>The spheroidal wave functions for integer r</u>.

It will be noted from (9.3.4) and (9.3.5) that $y_1(z) = y_2(z-\pi)$, so that the oblate spheroidal wave functions

$$A(\cos z) = \frac{1}{(\sin^2 z)^{1/4}}\left\{ y_1(z) \pm y_2(z) \right\}, \tag{1}$$

are either even or odd with respect to $z = \frac{1}{2}\pi$. Alternatively, $y_1(\cos z) = y_2(-\cos z)$ and (1) therefore represents functions which are either even or odd functions of $\cos z$. Following the notation of Meixner and Schäfke (1954) (see also Müller 1960), we denote these functions by $\overline{PS}^{\,r}_{r+n}$ where n is the number of zeros in the interval $0 \le z \le \pi$. In this range, the even functions must have an even number of zeros signified by $n = n_e$, say, while the odd functions must have an odd number of zeros and for this case we write $n = n_o$. Thus,

$$\left.\begin{array}{l} \overline{Ps}^{\,r}_{r+n_e} \\[2ex] \overline{Ps}^{\,r}_{r+n_o} \end{array}\right\} = \frac{N}{(\sin^2 z)^{\frac{1}{4}}} \left\{ y_1(z) \pm y_2(z) \right\} , \qquad (2)$$

where N is a normalisation constant yet to be determined.

As $h \to \infty$, the number of zeros in the interval $0 \leq z \leq \pi$ of the spheroidal wave function $\overline{Ps}^{\,r}_{r+n}$ with r and n integral tend to coincidence with the $\frac{1}{2}(m_o - 2 - 2r)$ zeros of the generalised Laguerre function $L^{r}_{\frac{1}{4}(m_o-2-2r)}(1|x)$, as has been shown by e.g. Meixner and Schäfke (1954, § 3.92). (This incidentally also follows from the results of the preceding section.) It is thus apparent that for integer n_e and n_o

$n_e = \frac{1}{2}(m_o - 2 - 2r)$, where $m_o = 2r+2, 2r+6, 2r+10, \dots$

and

$n_o = \frac{1}{2}(m_o - 2 - 2r)$, where $m_o = 2r+4, 2r+8, 2r+12, \dots$.

To make the oblate spheroidal wave functions unique, certain boundary conditions must be imposed. These are (cf. Meixner and Schäfke 1954, § 3.21; Erdélyi et al. 1955, § 16.11)

$$\overline{ps}^{\,r}_{r+n_o}\left(z=\tfrac{1}{2}\pi\right)=0\ , \tag{3}$$

$$\left\{\frac{\partial}{\partial z}\,\overline{ps}^{\,r}_{r+n_e}(z)\right\}_{z=\frac{1}{2}\pi}=0\,. \tag{4}$$

From the expansions of section 9.3, which are valid for $z \simeq \frac{1}{2}\pi$, it will be seen that (3) and (4) are already satisfied by (2). Thus, in contrast with the case for the Mathieu functions, no condition relating m to m_o can be found. It furthermore appears that this difficulty cannot be resolved by considering boundary conditions at $z=0$, where the results of § 9.5 apply, since by Sips (1949) the limit $z \to 0$ of $\overline{ps}^{\,r}_{r+n}(z)/(\sin^2 z)^{r/2}$ cannot vanish. In numerical applications therefore, m will have to be taken equal to m_o . In analogy with the results for Mathieu functions, this assumption is probably reasonable when h is large.

It is customary to adopt the normalisation condition (cf. Meixner and Schäfke 1954, § 3.23; Erdélyi et al. 1955, § 16.11)

$$\int_{-1}^{1} \left\{ \overline{ps}_{r+n}^{\,r} (v = \cos z, h) \right\}^2 dv = \frac{(2r+n)!}{(h+r+\frac{1}{2})\, n!} \,. \tag{5}$$

This yields an expansion for N which is just equal to that for $\overline{N}_A^{\,r}(m, h)$ given by (5.58) in Müller (1960). Finally, therefore,

$$N = \left\{ \frac{2 \quad \{\frac{1}{2}(m-2+2r)\}!}{(m-1) \quad \{\frac{1}{2}(m-2-2r)\}!\, \{\frac{1}{4}(m-2+2r)\}!\, \{\frac{1}{4}(m-2-2r)\}!} \right\}^{1/2}$$

$$\times (-1)^{\frac{1}{4}(m-2-2r)} (8h)^{m/4} e^{-2h} \left[1 - \frac{1}{2^7 h} (3m^2 + 4 - 4r^2) \right.$$

$$+ \frac{1}{2^{15} h^2} \left\{ 7m^4 - 80m^3 - 24m^2 - 320m - 16 - 16r^4 \right.$$
$$\left. - 8r^2(m^2 - 24m - 4) \right\}$$

$$- \frac{1}{3 \cdot 2^{22} h^3} \left\{ 9m^6 - 528m^5 + 4932m^4 + 3840m^3 \right.$$

$$+ 43056 m^2 + 11520 m + 18624 + 320 r^6$$

$$- 16 r^4 (21 m^2 - 48 m - 356)$$

$$+ 4 r^2 (15 m^4 + 288 m^3 - 4152 m^2 - 3072 m - 6160) \Big\}$$

$$+ O(h^{-4}) \Big] ; \qquad (6)$$

10. EXPRESSIONS FOR THE LATE TERMS IN THE GREEN-TYPE EXPANSIONS OF MODIFIED BESSEL FUNCTIONS.

10.1 Introduction.

In earlier chapters, direct application of an iterative procedure to 'the integro-differential equation' yielded the first few contributions V_i to its solution V as a series in inverse powers of a large parameter. To achieve good accuracy for V, it is characteristically only necessary to sum these leading contributions; yet for certain combinations of parameter(s) and variable stopping at early terms may not give sufficiently accurate results. Under these circumstances, late terms need to be calculated in preparation of the extended Borel method (1899) for summing a divergent series, developed and exemplified by Dingle in a number of papers (1958a,b,c; 1959a,b,c).

The iterative method referred to above is of no use for this purpose as it does not provide the general term in V_i required for Dingle's technique. However, reference to a typical integro-differential equation, (4.2.1) say, shows that for a given v_i the ratio of the contribution from the differential part to that from the integral part is roughly proportional to i^2. Therefore, when i is large, a reasonable approximation to v is obtained by taking only the first few v_i in the integral part. There results a first-order inhomogeneous differential equation, the solution of which can be expressed as an integral by the introduction of an integrating factor. This integral could be evaluated by repeated integration by parts thus yielding an approximate expression for the leading terms in v_i for any i. On the other hand, it is also possible to derive an exact expression for these terms by the introduction of a new variable whose form is suggested by the integrating factor already mentioned. In the present chapter the latter procedure is applied to modified Bessel functions, while in the succeeding chapter this method is applied to parabolic cylinder functions.

10.2 Preliminary considerations.

Only the calculations for the function denoted in chapter 3 by v will be given here. Corresponding results for the second solution, u , are easily deduced therefrom by means of relation (3.2.11).

The integro-differential equation for v is by (4.1.9)

$$2pv = q^2(q^2-1)v' + \frac{1}{4}\int_0^q (5q^2-1)v\,dq, \tag{1}$$

and in iterative form

$$2pv_{i+1} = q^2(q^2-1)v_i' + \frac{1}{4}\int_0^q (5q^2-1)v_i\,dq. \tag{2}$$

As already remarked in the introduction, an approximate expression for v may be obtained on replacing v in the integrand of (1) with the sum of the first few v_i as given by (4.2.2). Denoting the resulting integral by $f(q)$, (1) can be written as

$$2pv = q^2(q^2-1)v' + f(q) , \tag{3}$$

a solution of which is

$$v(z) = e^{2pz} \int_z^{\infty} e^{-2pz} f(q)\, dz \, . \tag{4}$$

Here, e^{-2pz} is the integrating factor and z is given by

$$z = \int^q \frac{dq}{q^2(q^2-1)} = \frac{1}{q} - \frac{1}{2} \ln \left| \frac{1+q}{1-q} \right| . \tag{5}$$

As has been pointed out, the integral in (4) may be evaluated by repeated integrations by parts leading to approximate results. Exact results are obtained, however, by assuming a power series expansion for V_i in terms of z .

10.3 <u>The power series method</u>.

To apply a power series method to (10.2.3), it is required to expand z in powers of q . This necessitates considering the cases q^2 greater than and less than unity.

For $q^2 > 1$, we have by (10.2.5) that

$$z = \frac{1}{q} - \coth^{-1} q = \frac{-1}{3q^3}\left(1 + \frac{3}{5q^2} + \frac{3}{7q^4} + \frac{3}{9q^6} + \cdots\right). \qquad (1)$$

This relation implies that q can be expanded in powers of $(3z)^{-1/3}$. Now V_i is a power series in q so that it is to be expected that $V_i(z)$ will be a power series in $(3z)^{-1/3}$. For the present case $(q^2 > 1)$, the leading term in $V_i(z)$ will be that giving rise to the term of highest power in q, which by (4.2.2) is q^{3i}. The coefficient of q^{3i} can be found exactly, and follows simply from the solution of

$$2pV = q^4 V' + \frac{5}{4}\int_0^q q^2 V \, dq, \qquad (2)$$

which is obtained from the integro-differential equation (10.2.1) by ignoring all terms except those which can produce contributions to the coefficient of q^{3i} in V_i. The differential form of (2) is

$$q^2 v'' + \left(4q - \frac{2p}{q^2}\right) v' + \frac{5}{4} v = 0. \tag{3}$$

It was shown in § 4.3 that the Mellin-Barnes integral representation for a solution of (3) is

$$v = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (\mu-1)!\,\left(-\mu-\frac{1}{6}\right)!\,\left(-\mu-\frac{5}{6}\right)!\,\left(-\frac{3q^3}{2p}\right)^{-\mu} d\mu. \tag{4}$$

The required leading term in $v_i(q)$ is therefore

$$\frac{\left(i-\frac{5}{6}\right)!\left(i-\frac{1}{6}\right)!}{i!} \left(\frac{3q^3}{2p}\right)^{i}. \tag{5}$$

Moreover, the lower limit of integration in (10.2.2) was chosen such that v_i is either exactly even or exactly odd depending on whether i is even or odd (cf. § 4.2). In view of these arguments, it is thus expected that $V_i(z)$ may be expanded as

$$V_i(z) = \frac{1}{2\pi} \frac{\left(i-\frac{5}{6}\right)!\left(i-\frac{1}{6}\right)!}{i!\,(-2pz)^i} \sum_{r=0}^{i} A_r(i)\,(3z)^{\frac{2}{3}r}, \tag{6}$$

where $A_r(i)$ are coefficients to be determined, and the factor $1/2\pi$ has been included to comply with the condition $V_0 = 1$ - this implies incidentally that $A_0(i) = 1$. The upper limit of summation in (6) has been put equal to i to ensure that the series terminates at q^i, a requirement which follows at once from (4.2.2).+

For $q^2 < 1$, expansion of the right-hand side of (10.2.5) yields

$$z = \frac{1}{q} - \tanh^{-1} q = \frac{1}{q}\left(1 - q^2 - \frac{1}{3}q^4 - \frac{1}{5}q^6 - \frac{1}{7}q^8 \cdots\right). \tag{7}$$

Now, V_i is expressible as a power series in z and the dominant term will be that giving rise to the term of lowest, or i th, power in q. Collecting together the terms in (10.2.1) which contribute to the coefficient of q^i, we see that the leading term follows from the series solution of

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\+ As will be shown in the next section, $A_r(i) = 0$ for $r > i$ so that we might just as well have taken the upper limit of summation as $r = \infty$.

$$2pv = -q^2 v' - \frac{1}{4}\int_0^q v\,dq\,, \tag{8}$$

or of

$$q^2 v'' + 2(p+q)v' + \frac{1}{4} v = 0\,. \tag{9}$$

Assuming a solution

$$v = \sum_{r=0}^{\infty} a_r q^r\,, \tag{10}$$

the method of Frobenius (cf. ch.2) yields

$$\frac{a_{r+1}}{a_r} = -\frac{(r+\frac{1}{2})^2}{2p(r+1)}\,. \tag{11}$$

Thus

$$v = \frac{1}{\pi}\sum_{r=0}^{\infty}\frac{\{(r-\frac{1}{2})!\}^2}{r!}\left(-\frac{q}{2p}\right)^r, \tag{12}$$

where the factor $1/\pi$ comes in through the condition

$V_0 = 1$. In view of the above, we are therefore led to assume the following expansion for $V_i(z)$

$$V_i(z) = \frac{1}{\pi}\,\frac{\left\{(i-\frac{1}{2})!\right\}^2}{i!\,(-2pz)^i}\,\sum_{r=0}^{i} B_r(i)\, z^{-2r}, \qquad (13)$$

where the upper limit of summation has been chosen+ such that the highest power of q in $V_i(q)$ is $3i$, and $B_r(i)$ are coefficients which remain to be determined. The condition $V_0 = 1$ implies that $B_0(i)=1$.

10.4 Determination of the coefficients $A_r(i)$.

Let expansion (10.3.6) be written explicitly

$$2\pi V_i(z) = \frac{(i-\frac{5}{6})!\,(i-\frac{1}{6})!}{i!\,(-2pz)^i}\Big[1 + A_1(i)(3z)^{\frac{2}{3}} + A_2(i)(3z)^{\frac{4}{3}} + A_3(i)(3z)^2 + A_4(i)(3z)^{\frac{8}{3}} + A_5(i)(3z)^{\frac{10}{3}} + \dots\Big]. \qquad (1)$$

+ See, however, the footnote in connexion with $A_r(i)$.

To determine $A_r(i)$, express the right-hand side of (1) as a series in q with the aid of (10.3.1) and then substitute for V_i in (10.2.2). Equating the coefficients of equal powers in q , we obtain that $A_1(i) = 0$, and also that $A_2(i)$ satisfies the following two-term recurrence relation

$$\frac{(i-\frac{1}{3})(i+\frac{1}{6})(i+\frac{5}{6})}{(i+1)} A_2(i+1) - (i-\frac{7}{6})(i-\frac{1}{2}) A_2(i) = -\frac{1}{315} . \tag{2}$$

This difference equation is solved most conveniently by writing

$$A_2(i) = \overset{\circ}{A}_2(i)\, m_2(i), \tag{3}$$

where $\overset{\circ}{A}_2(i)$ satisfies (2) with the right-hand side equal to zero, i.e.

$$\frac{\overset{\circ}{A}_2(i+1)}{\overset{\circ}{A}_2(i)} = \frac{(i-\frac{7}{6})(i-\frac{1}{2})(i+1)}{(i-\frac{1}{3})(i+\frac{1}{6})(i+\frac{5}{6})} . \tag{4}$$

The solution of this equation is

$$A_2^{\circ}(i) = \frac{(i-\frac{3}{2})!\ \ i!}{(i-\frac{1}{6})(i-\frac{7}{6})\,(i-\frac{4}{3})!\,(i-\frac{5}{6})!} \; . \tag{5}$$

Substitution of (3) into (2) now leads to the sum-equation

$$m_2(i) = -\frac{1}{315} \mathop{\mathsf{S}} (i-\tfrac{1}{6}) \frac{(i-\frac{4}{3})!\,(i-\frac{5}{6})!}{i!\ \ (i-\frac{1}{2})!} \; , \tag{6}$$

where S is the summation operator defined in e.g. Nörlund (1923, pp. 40-47). The sum can be performed by a method outlined in § 4.3 and we obtain

$$m_2(i) = -\frac{1}{105} \; \frac{(i-\frac{4}{3})!\,(i-\frac{5}{6})!}{(i-\frac{3}{2})!\ (i-1)!} \; . \tag{7}$$

Combination of (7) with (3) and (5) yields

$$A_2(i) = -\frac{1}{105} \; \frac{i}{(i-\frac{1}{6})(i-\frac{7}{6})} \; . \tag{8}$$

With the aid of (8), (1) and (10.3.1) we can already derive the exact coefficients of q^{3i}, q^{3i-2} and q^{3i-4} in $V_i(q)$ for any i. That (8) is indeed correct may be verified by calculating the coefficient of q^{3i-4} in $V_i(q)$ for $i = 5$, say. Substitution in (1) of expansion (10.3.1) for z yields, for $i = 5$,

$$u = \frac{1}{2\pi} \frac{\left(\frac{25}{6}\right)! \left(\frac{29}{6}\right)!}{5! \ (2p)^5} \left(3q^3\right)^5 \left(1 + \frac{3}{5q^2} + \frac{3}{7q^4} + \cdots\right)^{-5}$$

$$\times \left[1 + A_2(5) \frac{1}{q^4}\left(1 + \frac{4}{5q^2} + \cdots\right) + \frac{A_3(5)}{q^6}\left(1 + \cdots\right)\right], \quad (9)$$

giving $\dfrac{108\ 313\ 205}{1\ 179\ 648\ p^5}$ as the coefficient of q^{11}, in agreement with the corresponding coefficient quoted in (4.2.2). Subsequent results may be checked by an analogous procedure.

Similarly, it may be shown that $A_3(i)$ satisfies the equation

$$\frac{\left(i+\frac{1}{6}\right)\left(i+\frac{5}{6}\right)(i-1)}{(i+1)} A_3(i+1) - \left(i-\frac{7}{6}\right)\left(i-\frac{11}{6}\right) A_3(i) = -\frac{2}{675} \cdot \quad (10)$$

Writing

$$A_3(i) = \overset{\circ}{A}_3(i)\, m_3(i), \quad (11)$$

substitution into (10) leads to

$$\overset{\circ}{A}_3(i) = \frac{i(i-1)}{\left(i-\frac{1}{6}\right)\left(i-\frac{7}{6}\right)\left(i-\frac{5}{6}\right)\left(i-\frac{11}{6}\right)}, \quad (12)$$

and

$$m_3(i) = -\frac{2}{675} \mathop{\mathsf{S}} \frac{\left(i-\frac{1}{6}\right)\left(i-\frac{5}{6}\right)}{i(i-1)}$$

$$= -\frac{2}{675} \mathop{\mathsf{S}} \left\{1 + \frac{5}{36}\frac{1}{i(i-1)}\right\}. \quad (13)$$

The summation in (13) is easily carried out giving

$$m_3(i) = -\frac{2}{675}\left\{ i - \frac{5}{36(i-1)} + C \right\}, \tag{14}$$

where C is a summation constant which will now be determined to achieve agreement with the Green-type expansion of § 4.2. $A_3(i)$ in (1) contributes to the coefficient of q^{3i-6} in $V_i(q)$. We therefore take $i = 3$ and claculate the coefficient of q^3/p^3 from (1), choosing C such that this coefficient is equal to that given in (4.2.2). This yields $C = -1$, so that the complete expression for $A_3(i)$ is by (11), (12) and (14)

$$A_3(i) = -\frac{i\left\{36(i-1)^2 - 5\right\}}{12150\,(i-\frac{1}{6})(i-\frac{7}{6})(i-\frac{5}{6})(i-\frac{11}{6})}. \tag{15}$$

Routine, but rather tedious, calculation shows that the difference equation for $A_4(i)$ is

$$\frac{(i+\frac{1}{6})(i+\frac{5}{6})(i-\frac{5}{3})}{(i+1)} A_4(i+1) - (i-\tfrac{11}{6})(i-\tfrac{5}{2})\,A_4(i) =$$

$$-\frac{23}{40425} + \frac{i}{33075\,(i-\frac{1}{6})(i-\frac{7}{6})}. \tag{16}$$

With the right-hand side of (16) put equal to zero, a solution, $A_4^{\circ}(i)$ say, is

$$A_4^{\circ}(i) = \frac{(i-\frac{7}{2})!\; i!}{(i-\frac{5}{6})(i-\frac{11}{6})(i-\frac{1}{6})!\,(i-\frac{8}{3})!}\,, \tag{17}$$

and if we write

$$A_4(i) = A_4^{\circ}(i)\, m_4(i) \tag{18}$$

the sum-equation for $m_4(i)$ becomes

$$m_4(i) = -\frac{23}{40425} \mathop{S} (i-\tfrac{5}{6}) \frac{(i-\frac{8}{3})!\,(i-\frac{1}{6})!}{(i-\frac{5}{2})!\;\; i!}$$

$$+\frac{1}{33075} \mathop{S} (i-\tfrac{5}{6}) \frac{(i-\frac{8}{3})!\,(i-\frac{13}{6})!}{(i-\frac{5}{2})!\,(i-1)!}\,. \tag{19}$$

The summations in (19) require no new techniques, and $A_4(i)$ is derived in a straightforward manner.

Including $A_4(i)$, the expression for $v_i(z)$ becomes finally

$$V_i(z) = \frac{1}{(i-1)!\,(-54pz)^i}\Bigg[\frac{(3i-\frac{1}{2})!}{i(i-\frac{1}{2})!}$$

$$-\frac{3}{35}\,\frac{(3i-\frac{5}{2})!}{(i-\frac{7}{6})(i-\frac{3}{2})!}\,(3z)^{\frac{4}{3}}$$

$$-\frac{2\cdot 3^3}{25}\,\frac{(3i-\frac{13}{2})!}{(i-\frac{5}{2})!}\left\{(i-1)^2-\frac{5}{36}\right\}(3z)^2$$

$$-\frac{3^4}{269500}\,\frac{(3i-\frac{13}{2})!}{(i-\frac{5}{2})!}\,(828i^2-1214i+271)$$

$$\times\,(3z)^{\frac{8}{3}}$$

$$+\;\cdots\Bigg]\;,\qquad (q^2>1)\,, \tag{20}$$

where a contraction was achieved by means of the triplication formula for factorial functions

$$(3n)! = \frac{3^{\frac{1}{2}+3n}}{2\pi}\, n!\,(n-\tfrac{1}{3})!\,(n-\tfrac{2}{3})! . \qquad (21)$$

10.5 Determination of the coefficients $B_r(i)$.

The coefficients $B_r(i)$ are calculated by the same method as that used for the determination of $A_r(i)$, except that now z is replaced with the series for $q^2 < 1$ given by (10.3.7).

Substituting (10.3.13) into (10.2.2) and equating the coefficients of q^{i+3}, we are led to the following difference equation for $B_1(i)$:

$$\frac{(i+\frac{1}{2})^2(i+3)}{(i+1)} B_1(i+1) - (i+\tfrac{5}{2})^2 B_1(i) = -2 . \qquad (1)$$

If $B_1^0(i)$ denotes the solution of the homogeneous part of (1), then

$$B_1^0(i) = \frac{(i+\frac{3}{2})^2(i+\frac{1}{2})^2}{(i+1)(i+2)} . \qquad (2)$$

Also, writing

$$B_1(i) = \overset{\circ}{B}_1(i)\, n_1(i), \tag{3}$$

and substituting into (1), we derive that

$$n_1(i) = -2 \mathop{\large S} \frac{(i+1)(i+2)}{(i+\frac{1}{2})^2(i+\frac{3}{2})^2(i+\frac{5}{2})^2}. \tag{4}$$

To perform the summation in (4) it is expedient to separate the summand into partial fractions as follows

$$\frac{(i+1)(i+2)}{(i+\frac{1}{2})^2(i+\frac{3}{2})^2(i+\frac{5}{2})^2} = \frac{1}{3}\frac{(i+1)}{(i+\frac{1}{2})^2(i+\frac{3}{2})^2}$$

$$- \frac{1}{3}\frac{(i+2)}{(i+\frac{3}{2})^2(i+\frac{5}{2})^2} + \frac{1}{12}\frac{1}{(i+\frac{1}{2})^2(i+\frac{3}{2})^2(i+\frac{5}{2})^2}. \tag{5}$$

The first two terms on the right-hand side of (5) are easily summed since

$$\mathop{\large S} \frac{(i+1)}{(i+\frac{1}{2})^2(i+\frac{3}{2})^2} = -\frac{1}{2}\frac{1}{(i+\frac{1}{2})^2}, \tag{6}$$

as may be verified by applying the difference operator to both sides. The third term in (5) is summed by first re-expressing it as

$$\frac{1}{(i+\frac{1}{2})^2(i+\frac{3}{2})^2(i+\frac{5}{2})^2} = \frac{1/4}{(i+\frac{1}{2})^2} - \frac{3/4}{(i+\frac{1}{2})} + \frac{1/4}{(i+\frac{3}{2})^2}$$

$$+ \frac{1/4}{(i+\frac{5}{2})^2} + \frac{3/4}{(i+\frac{5}{2})} \, . \qquad (7)$$

The sum of each term on the right-hand side of (7) can be expressed in terms of the ψ - function (the logarithmic derivative of the factorial function) or its first derivative.[+] In the notation adopted here, the ψ - function and its derivatives are represented in summation form by (cf. Nörlund 1954, p.105)

$$\psi^{(n)}(i-\tfrac{1}{2}) = (-1)^n \, n! \, \sum \frac{1}{(i+\frac{1}{2})^n} + \text{constant}. \qquad (8)$$

The complete sum in (4) can now be derived and invoking

[+] In most older texts, these functions are generally referred to as the di- and trigamma functions (cf.,e.g., Davis 1935, Jordan 1960 reprint).

(3) this yields

$$B_1(i) = \frac{-i(4i^2+16i+11)}{16(i+1)(i+2)}$$

$$+\frac{1}{4}\frac{(i+\frac{3}{2})^2(i+\frac{1}{2})^2}{(i+1)(i+2)}\left\{\psi'(i-\tfrac{1}{2}) + C\right\}, \qquad (9)$$

where the constant C is determined by substituting for $B_1(i)$ in (1) and employing the relation

$$\psi'(i-\tfrac{1}{2}) = \tfrac{1}{2}\pi^2 - 4\left(\frac{1}{1^2}+\frac{1}{3^2}+\cdots+\frac{1}{(2i-1)^2}\right). \qquad (10)$$

This gives $C = -\frac{1}{2}\pi^2$.

The calculation for $B_2(i)$ is relatively straightforward but extremely laborious. We quote here only the main results. The difference equation obtained by methods already discussed is

$$\frac{(i+\frac{1}{2})^2(i+5)}{(i+1)} B_2(i) - \left(i+\frac{9}{2}\right)^2 B_2(i) =$$

$$-\frac{1}{2}\frac{(i+\frac{3}{2})^2(i+\frac{1}{2})^2}{(i+1)(i+2)}\left\{\psi'(i-\tfrac{1}{2}) - \tfrac{1}{2}\pi^2\right\}$$

$$+ \quad \frac{12\,i^3 + 248\,i^2 + 633\,i + 400}{24\,(i+1)\,(i+2)} \quad . \tag{11}$$

The solution of the homogeneous part of (11), $B_2^0(i)$ say, is

$$B_2^0(i) = \frac{\left\{(i+\frac{7}{2})!\right\}^2 \, i!}{\left\{(i-\frac{1}{2})!\right\}^2 (i+4)!} \quad . \tag{12}$$

If now we write

$$B_2(i) = B_2^0(i)\, n_2(i) \, , \tag{13}$$

substitution in (11) leads to

$$n_2(i) = -\frac{1}{2} \sum \frac{(i+3)(i+4)}{(i+\frac{5}{2})^2 (i+\frac{7}{2})^2 (i+\frac{9}{2})^2} \times \left\{ \psi'(i-\tfrac{1}{2}) - \tfrac{1}{2}\pi^2 \right\}$$

$$+ \sum \frac{(12\,i^3 + 248\,i^2 + 633\,i + 400)(i+3)(i+4)}{24(i+\frac{1}{2})^2 (i+\frac{3}{2})^2 (i+\frac{5}{2})^2 (i+\frac{7}{2})^2 (i+\frac{9}{2})^2} \, . \tag{14}$$

The second summation is merely tedious and can be performed by separating the summand into partial fractions. The first summation is carried out by summation by parts, using the formula (cf. Jordan 1960 reprint, § 34)

$$\mathsf{S}\left\{ U(i) V(i) \right\} = U(i) \,\mathsf{S}\, V(i) - \mathsf{S}\left[\left\{ \mathsf{S}\, V(i+1) \right\} \Delta U(i) \right], \qquad (15)$$

where Δ is the differencing operator. We choose

$$\left. \begin{aligned} U(i) &= \psi'(i - \tfrac{1}{2}) - \tfrac{1}{2}\pi^2, \\ V(i) &= \frac{(i+3)(i+4)}{(i+\frac{5}{2})^2 (i+\frac{7}{2})^2 (i+\frac{9}{2})^2}. \end{aligned} \right\} \qquad (16)$$

$\mathsf{S}\, V(i)$ is carried out in the usual way by separating $V(i)$ into its partial fractions. The second summation on the right of (15) involves

$$\mathbf{S} \frac{\{\psi'(i+\frac{1}{2}) - \frac{1}{2}\pi^2\}}{(i+\frac{1}{2})^2} \tag{17}$$

This sum is itself performed by application of (15): by definition (8)

$$\psi'\left(i+\tfrac{1}{2}\right) = \psi'\left(i-\tfrac{1}{2}\right) - \frac{1}{(i+\frac{1}{2})^2}, \tag{18}$$

and hence

$$\mathbf{S} \frac{\{\psi'(i+\frac{1}{2}) - \frac{1}{2}\pi^2\}}{(i+\frac{1}{2})^2} = -\mathbf{S} \frac{1}{(i+\frac{1}{2})^4} + \mathbf{S} \frac{\{\psi'(i-\frac{1}{2}) - \frac{1}{2}\pi^2\}}{(i+\frac{1}{2})^2}. \tag{19}$$

Furthermore, if in (15) we take

$$U(i) = \psi'(i-\tfrac{1}{2}) - \tfrac{1}{2}\pi^2, \tag{20}$$

and

$$V(i) = \frac{1}{(i+\frac{1}{2})^2}, \tag{21}$$

it is evident that

$$\sum \frac{\left\{\psi'(i-\tfrac{1}{2})-\tfrac{1}{2}\pi^2\right\}}{(i+\tfrac{1}{2})^2} = -\left\{\psi'(i-\tfrac{1}{2})-\tfrac{1}{2}\pi^2\right\}^2$$

$$-\sum \frac{\left\{\psi'(i+\tfrac{1}{2})-\tfrac{1}{2}\pi^2\right\}}{(i+\tfrac{1}{2})^2}. \qquad (22)$$

Combination of (19) and (22) therefore yields the required sum (17), i.e.

$$\sum \frac{\left\{\psi'(i+\tfrac{1}{2})-\tfrac{1}{2}\pi^2\right\}}{(i+\tfrac{1}{2})^2} = -\frac{1}{2}\sum \frac{1}{(i+\tfrac{1}{2})^4}$$

$$-\frac{1}{2}\left\{\psi'(i-\tfrac{1}{2})-\tfrac{1}{2}\pi^2\right\}^2. \qquad (23)$$

The remaining summations do not require any other new techniques, and (12), (13) and (14) eventually yield

$$B_2(i) = \frac{\left\{(i+\tfrac{7}{2})!\right\}^2 i!}{32\,(i+4)!\left\{(i-\tfrac{1}{2})!\right\}^2}\left[\left\{\psi'(i-\tfrac{1}{2})-\tfrac{1}{2}\pi^2\right\}^2 + \frac{1}{6}\left\{\psi'''(i-\tfrac{1}{2})-\pi^4\right\}\right]$$

$$- \left(64i^7 + 1024i^6 + 6672i^5 + 23552i^4 + 49388i^3 + 61888i^2 + 43171i + 13312\right)$$

$$\times \frac{i!\left\{\psi'(i-\tfrac{1}{2}) - \tfrac{1}{2}\pi^2\right\}}{1024\ (i+4)!}$$

$$+ \frac{i\left(144i^5 + 2256i^4 + 13848i^3 + 37320i^2 + 39873i + 9681\right) i!}{4608\ (i+4)!} \, . \tag{24}$$

For $q^2 < 1$, equations (10.3.7), (10.3.13) and (24) now give the three leading terms in $V_i(q)$ exactly for any i, and approximations to terms of higher power in q. It may be noted, incidentally, that since $A_r(i)$ has been calculated up to $r = 3$ and $B_r(i)$ up to $r = 2$, (10.3.6) and (10.3.13) can be used to give $V_i(q)$ exactly up to $i = 7$.

11. LATE TERMS IN THE GREEN-TYPE EXPANSIONS OF PARABOLIC CYLINDER FUNCTIONS.

11.1 The integrating factor.

The methods of chapter 10 are now applied to parabolic cylinder functions. Once again it is only necessary to derive results for V as defined in § 5.2, since corresponding results for u follow from (3.2.11).

The integro-differential equation for V is by (5.2.6)

$$a^2 V = (\alpha^2-1)^2 V' + \frac{1}{4}\int^{\alpha} (5\alpha^2-2) V \, d\alpha , \qquad (1)$$

which can be written in iterative form as

$$a^2 V_{i+1} = (\alpha^2-1)^2 V_i' + \frac{1}{4}\int^{\alpha} (5\alpha^2-2) V_i \, d\alpha . \qquad (2)$$

An approximation to V is obtained on replacing V

in the integral part of (1) by the sum of the first few v_i . Denoting the resulting integral by $f(\alpha)$, (1) becomes approximately

$$a^2 v = (\alpha^2 - 1)^2 v' + f(\alpha) . \tag{3}$$

A solution of (3) is

$$v = -e^{a^2 z} \int^{z} e^{-a^2 z} f(\alpha)\, dz , \tag{4}$$

where z in the integrating factor $e^{-a^2 z}$ is given by

$$z = \frac{1}{2} \left\{ \frac{\alpha}{1-\alpha^2} + \frac{1}{2} \ln \left| \frac{1+\alpha}{1-\alpha} \right| \right\} . \tag{5}$$

11.2 <u>The power series method</u>.

The power series method introduced in chapter 10 can be applied when the series in α for z is known. Reference to (11.1.5) shows that this requires consideration of the two cases $\alpha^2 > 1$ and $\alpha^2 < 1$.

For $\alpha^2 > 1$,

$$z = \frac{-1}{3\alpha^3}\left[1 + \frac{6}{5\alpha^2} + \frac{9}{7\alpha^4} + \frac{4}{3\alpha^6} + \frac{15}{11\alpha^8} + \cdots\right], \qquad (1)$$

implying that α is expressible as a power series in $(3z)^{-1/3}$. α^{3i} is the highest power of α in V_i ; its coefficient can be found exactly from the solution of

$$a^2 V = \alpha^4 V' + \frac{5}{4}\int^{\alpha} \alpha^2 V \, d\alpha , \qquad (2)$$

as these are the only factors in (11.1.1) that contribute in this case. Now (2) is just equal to (10.3.2) with $\rho = \frac{1}{2} a^2$, and hence by (10.3.5) the required leading term is

$$\frac{(i-\frac{5}{6})!\,(i-\frac{1}{6})!}{i\,!}\left(\frac{3\alpha^3}{a^2}\right)^i . \qquad (3)$$

In sections 5.3 and 6.11 we quoted two possible expansions for $V_i(\alpha)$. The expansion (6.11.6) was obtained as a special case of the Green-type expansion for the Whittaker function $W_{k,m}(z)$ given by (6.1.20). The contributions $V_i(\alpha)$ of section 5.3 were derived directly from (5.3.1) with the lower limit of integration selected such that no constant terms appeared in $V_{i\neq 0}(\alpha)$, i.e. here $V_{i\neq 0}(\alpha=0)=0$. In the present chapter, the terms $V_i(\alpha)$ will be identified with those calculated by the latter scheme, and will therefore correspond to those recorded in (5.3.2). The expansion for V_i in terms of z thus takes the form

$$V_i(z) = \frac{1}{2\pi}\,\frac{(i-\frac{5}{6})!\,(i-\frac{1}{6})!}{i!\,(-a^2 z)^i}\sum_{r=0}^{\infty} A_r(i)\,(3z)^{\frac{2}{3}r}, \qquad (4)$$

where the factor $1/2\pi$ has been included to satisfy the conditions $V_0=1$ and $A_0(i)=1$. As will be seen in the succeeding section, the coefficients $A_r(i)$ ensure that the expansion for $V_i(\alpha)$ terminates at α or α^2, depending on whether i is odd or even.

The case $\alpha^2 < 1$ has not yielded to treatment by our methods. The fundamental difficulty arising here is that any inclusion of the term $(-1)^2 V_i'$ in (11.1.2) presupposes that the coefficients of all powers of α for smaller values of i are known. For example, a term α^r in $V_i(\alpha)$ will contribute to the coefficient of α in $V_{i+r-1}(\alpha)$ regardless of the magnitude of r. This difficulty cannot be resolved simply by putting $\alpha = 1 \pm q$, for then z becomes given by (cf. (11.1.5))

$$z = \frac{1}{2}\left\{\frac{1 \pm q}{(\mp q)(2 \pm q)} + \frac{1}{2}\ln\left|\frac{2 \pm q}{\mp q}\right|\right\}, \qquad (5)$$

and now z cannot be expressed as a power series in q, a prerequisite for the application of our method.

11.3 Determination of the coefficients $A_r(i)$.

Substitution of (11.2.4) into (11.1.2) yields with the aid of (11.2.1) that $A_1(i) = 0$, and that $A_2(i)$ satisfies the recurrence relation

$$\frac{(i+\frac{1}{6})(i+\frac{5}{6})(i-\frac{1}{3})}{(i+1)} A_2(i+1) - (i-\tfrac{1}{2})(i-\tfrac{7}{6}) A_2(i) = \frac{1}{140} . \tag{1}$$

By comparison with (10.4.2) and (10.4.8) therefore

$$A_2(i) = \frac{3}{140} \frac{i}{(i-\frac{1}{6})(i-\frac{7}{6})} . \tag{2}$$

Similarly, $A_3(i)$ satisfies the equation

$$\frac{(i+\frac{1}{6})(i+\frac{5}{6})(i-1)}{(i+1)} A_3(i+1) - (i-\tfrac{7}{6})(i-\tfrac{11}{6}) A_3(i) = -\frac{7}{1350} . \tag{3}$$

A solution of (3) is

$$A_3(i) = \frac{-i(252i^2 - 3204i + 2917)}{4860(i-\frac{1}{6})(i-\frac{7}{6})(i-\frac{5}{6})(i-\frac{11}{6})} . \tag{4}$$

$A_4(i)$ is most conveniently calculated by analogy with the calculation of $A_4(i)$ for the modified Bessel functions. Equation (10.4.16) suggests that $A_4(i)$ satisfies an equation of form

$$\frac{(i+\frac{1}{6})(i+\frac{5}{6})(i-\frac{5}{3})}{(i+1)} A_4(i+1) - (i-\tfrac{11}{6})(i-\tfrac{5}{2}) A_4(i)$$

$$= A + B\, A_2(i), \tag{5}$$

where A and B are constants and $A_2(i)$ is as in (2). The solution of (5) can be written as

$$A_4(i) = \frac{a\, i}{(i-\frac{1}{6})(i-\frac{7}{6})} \left\{ 1 + b\, \frac{(i-1)}{(i-\frac{5}{6})(i-\frac{11}{6})} \right\}. \tag{6}$$

Here the constants a and b are determined by identifying (11.2.4) with (5.3.2) via (11.2.1) at two convenient values of i, $i = 1$ and $i = 2$ say. This yields

$$a = \frac{453}{269500}, \quad b = \frac{5327}{3624}. \tag{7}$$

Finally therefore,

$$V_i(z) = \frac{1}{(i-1)!\,(-27\alpha^2 z)^i}\left[\frac{(3i-\frac{1}{2})!}{i\,(i-\frac{1}{2})!}\right.$$

$$-\frac{27}{140}\,\frac{(3i-\frac{5}{2})!}{(i-\frac{7}{6})(i-\frac{3}{2})!}\,(3z)^{\frac{4}{3}}$$

$$-\frac{3}{20}\,\frac{(3i-\frac{13}{2})!}{(i-\frac{5}{2})!}\,(252i^2-3204i+2917)(3z)^2$$

$$-\frac{81}{107\,8000}\,\frac{(3i-\frac{13}{2})!}{(i-\frac{5}{2})!}\,(10872i^2-13011i+629)(3z)^{\frac{8}{3}}$$

$$\left.+\cdots\right]. \tag{8}$$

This expression gives the first five terms in V_i for any i, and approximations to further terms provided i is large and $\alpha^2 > 1$.

12. CONCLUDING REMARKS, AND SOME SUGGESTIONS FOR FURTHER RESEARCH.

In the main body of this thesis, straightforward formal methods have been developed for deriving asymptotic expansions from certain differential equations which occur frequently in problems connected with physics and engineering. Our applications have covered a wide range of higher transcendental functions, including such important representatives of the class of confluent hypergeometric functions as modified Bessel functions, Weber parabolic cylinder functions, Whittaker functions and the general exponential integral; and also Mathieu functions and oblate spheroidal wave functions. In the course of our investigations several problems have suggested themselves, and we list the following topics as examples of problems for the solution of which our work may serve as a starting point:

(i) A general method should be derived for constructing an integral representation for the quantity which we have denoted by V. This would serve to establish a close correspondence between the Liouville-Green method and the method of steepest descents. The importance of such an investigation lies in the fact that expressions for the late terms in Green-type expansions (see (ii)) are generally determined more easily via the steepest descents results - i.e. via the integral representation of v - than by the Liouville-Green method (cf. Malaviya 1963, ch.7). If it were possible therefore to link up the Liouville-Green and steepest descents methods, the former could be employed for the first few terms with the latter furnishing the late terms in the Green-type expansion.

Tentative steps in this direction have already been taken for the modified Bessel functions. Adopting the notation of chapter 3, reference to Malaviya (1963, ch.4) shows that the integral representation for $y = \chi^{1/4} v$ may be expected to have the general form

$$y = \int_a^b e^{-\frac{p}{q}\left\{f(w) + q\,g(w)\right\}}\, W(w)\, dw, \qquad (1)$$

where p is a large parameter, q is a function involving p and the independent variable, x say, of the given differential equation, and $W(w)$ is some weighting factor. The differential equation satisfied by v is (3.2.9), and the corresponding equation for y becomes in terms of q

$$q_1^2\, y'' + \left(q_2 - 2\gamma^{\frac{1}{2}} q_1\right) y' - \frac{1}{2}\gamma^{-\frac{1}{2}}\gamma_1\, y = 0, \qquad (2)$$

where primes signify differentiation with respect to q, and $q_n = (d/dx)^n q$, $\gamma_n = (d/dx)^n \gamma$. _Assuming_ that for the modified Bessel functions $W(w) = 1$, substitution of (1) into (2) yields

$$-\frac{1}{2}\frac{\gamma_1}{\gamma^{\frac{1}{2}}}\int_a^b F(w,q)\,dw$$

$$+\frac{p}{q^2}\left(q_2 - 2q_1\gamma^{\frac{1}{2}} - \frac{2q_1^2}{q}\right)\int_a^b f(w)\,F(w,q)\,dw$$

$$+\frac{p^2 q_1^2}{q^4}\int_a^b f^2(w)\,F(w,q)\,dw = 0, \qquad (3)$$

where

$$F(w,q) = e^{-\frac{p}{q}\left\{f(w) + q\,g(w)\right\}} . \tag{4}$$

Now, for the modified Bessel functions (cf. § 4.1)

$$\gamma = e^{2x} + p^2 , \tag{5}$$

and if we take $q = p/(e^{2x}+p^2)^{1/2}$ as in (4.1.8), (3) reduces to

$$- \int_a^b F(w,q)\,dw + \left(2\frac{p}{q} - 1 - q^2\right) \int_a^b f(w)\,F(w,q)\,dw$$

$$+ \frac{p}{q}\left(1-q^2\right) \int_a^b f^2(w)\;F(w,q)\,dw = 0 . \tag{6}$$

At this stage we make the limiting assumption

$$f(w) = g'(w) , \tag{7}$$

and (6) now becomes

$$\frac{1}{q}\int_a^b \left\{ f^2(w) + 2f(w) \right\} F(w,q)\, dw$$

$$+ \int_a^b \left\{ p\, f(w)\, f'(w) - f(w) - 1 \right\} F(w,q)\, dw$$

$$- \left[e^{-\frac{p}{q}\left\{ f(w) + p\, g(w) \right\}} \right]_a^b = 0 . \tag{8}$$

If (1) with $W(w) = 1$ is to be the steepest descents integral, the leading term in the exponent of the last term in (8) is $O(w^2)$. Thus the term in square brackets in (8) can be made to vanish by taking the limits of integration to be $-\infty$ and ∞. Integration by parts now leads to the equations

$$\left.\begin{aligned} f''(w) &= f(w) + 1 , \\ f^2(w) + 2f(w) &= f'(w) , \end{aligned}\right\} \tag{9}$$

with solution

$$f(\omega) = \cosh \omega - 1 . \tag{10}$$

By (7) therefore

$$g(\omega) = \sinh \omega - \omega , \tag{11}$$

and hence (1) becomes

$$y = \int_{-\infty}^{\infty} e^{-\frac{p}{q}\left\{(\cosh \omega - 1) + q(\sinh \omega - \omega)\right\}} d\omega , \tag{12}$$

in agreement with the steepest descents result obtained from Watson (1944, ch. VIII), and given also by Malaviya (1963, § 4.1).

The above is of course only a plausibility argument, and a full discussion should not have to rely on a priori assumptions such as (7). It might be possible to overcome this particular difficulty by replacing $f(\omega)$ and $g(\omega)$ in (1) with a suitable power series in ω and deriving the coefficients in these expansions with the aid of (6). Once a rigorous treatment has been developed for the modified Bessel functions, application should be made to more complicated cases, for example to the parabolic cylinder, Poiseuille and Whittaker functions. This would be particularly useful for the last two functions as here the method of steepest descents runs into difficulties for large values of the parameter(s) (cf. Malaviya, 1963, § 1).

(ii) The method of chapter 10 for deriving late terms in Green-type expansions failed for $\alpha^2 < 1$ in chapter 11. This case should certainly be investigated more fully. In addition, the method should be applied to Poiseuille functions and Whittaker functions $W_{k,m}$, with $|k|$ large, and with $|k^2 - m^2|$ large.

(iii) Attempts should be made to apply the Green-Liouville-Mellin transform technique to such functions as prolate spheroidal wave functions periodic Lamé functions, ellipsoidal functions and elliptic-conal wave functions. Asymptotic expansions for these functions have already been given by Müller (1960), but our methods should lead to simpler results, and also to new expansions uniform with respect to the independent variable.

(iv) In the foregoing we have obtained expansions which are uniformly valid with respect to the <u>independent variable</u>, and methods should now be developed for deriving expansions which are uniformly valid with respect to the <u>parameter</u> (assumed large in the above) in the given differential equation.

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